PRACE IPPT · IFTR REPORTS 8/2006

J. J. Sławianowski, V. Kovalchuk,
B. Gołubowska, A. Martens, E. E. Rożko

DYNAMICAL SYSTEMS WITH INTERNAL DEGREES OF FREEDOM IN NON-EUCLIDEAN SPACES

INSTYTUT PODSTAWOWYCH PROBLEMÓW TECHNIKI
POLSKIEJ AKADEMII NAUK
WARSZAWA 2006

ISSN 0208-5658

Redaktor Naczelny:
doc. dr hab. Zbigniew Kotulski

Recenzent:
prof. dr hab. Ryszard Mrugała –
Instytut Fizyki UMK, Toruń

Praca wpłynęła do Redakcji 7 lipca 2006 r.

Instytut Podstawowych Problemów Techniki PAN
Nakład: 100 egz. Ark. wyd. 8,15
Oddano do druku we wrześniu 2006 r.

Druk i oprawa: Drukarnia Braci Grodzickich, Piaseczno, ul. Geodetów 47a

Contents

Introduction

The primary concept of Newton mechanics is that of the material point moving in three-dimensional Euclidean space. A good deal of the theory depends only on the affine sector of geometry. The metric structure becomes essential when constructing particular functional models of forces; the concepts of energy, work, and power (time rate of work) also depend in an essential way on the metric tensor. The Galilei relativity principle implies that, as a matter of fact, it is not three-dimensional Euclidean space but rather four-dimensional Galilean space-time that is a proper arena of mechanics. This space-time has relatively complicated structure, does not carry any natural four-dimensional metric tensor and fails to be the Cartesian product of space and time. There exists the absolute time, but the absolute space does not. In the sequel we concentrate on the other kind of problems, so the analysis of the subtle space-time aspects will be almost absent in our treatment. Newton theory becomes essentially realistic and viable when multiparticle systems are analyzed. It is just there where metrical concepts become almost unavoidable, because it is practically impossible to construct any realistic model of interparticle forces without the explicit use of the metric tensor. Extended bodies are described as discrete or continuous systems of material points. Their motion consists of that of the center of mass, i.e., translational motion and the relative motion of constituents with respect to the center of mass. The total configuration space may be identified with the Cartesian product of the physical space (translational motion) and the configuration space of relative motion. In many physical problems the structure of mutual interactions leads to certain hierarchy of degrees of freedom of the relative motion; in particular, some constraints may appear. The effective configuration space becomes then the Cartesian product of the physical space and some manifold of additional degrees of freedom. There are situations when this auxiliary manifold and the corresponding dynamics are postulated as something rather primary then derived from the multiparticle models. Usually the guiding hints are based on some symmetry principles. In this way the concept of internal degrees of freedom replaces that of relative motion. Sometimes it is a merely convenient procedure, but one can also admit something like essentially internal degrees of freedom not derivable from any multiparticle model. After all, the very concept of the material point is an abstraction of a small piece of matter. Why to reject a priori an abstraction of a small and particularly shaped piece of matter, thus, the material point with extra attached geometric objects, as something primary for mechanics? In this way the configuration space becomes as follows:

$$Q = M \times Q^{\text{int}},$$

the Cartesian product of physical space M (translational motion) and some internal configuration space Q_{int}. The usual d'Alembert procedure of deriving equations of motion is not then reliable, perhaps just essentially inadequate and should be replaced by something else, probably based on appropriate invariance assumptions.

In standard quantum theory there are quantities which have the status of essentially internal variables, e.g., spin. If M is not the Euclidean space but some general manifold, then the concept of essentially internal degrees of freedom becomes even more justified, as we shall see.

And just in connection with this, the last step of generalization: why $Q = M \times Q^{\text{int}}$? Perhaps any spatial point $x \in M$ has its own manifold of internal variables Q^{int}_x? Then the total configuration space would be

$$Q = \bigcup_{x \in M} Q^{\text{int}}_x.$$

In this way, the concept of fibre bundle appears as a most natural mathematical framework for describing internal degrees of freedom.

As mentioned, such concepts are particularly natural when M is a general manifold endowed with some geometric structure based on the affine connection, metric tensor, or both (interrelated or not). In any case, the primary mathematical concept underlying any model of internal degrees of freedom is a principal fibre bundle (Q, M, π), where M denotes the base manifold (physical space), Q is the total bundle manifold (configuration space), and $\pi : Q \to M$ is the bundle projection. We shall often use the following abbreviation for fibres: $Q_x = \pi^{-1}(x)$. Obviously, in the relativistic theory a more adequate formulation is one using the space-time manifold as a basis of the fibre bundle. The same is true in non-relativistic theory when the problems of Galilean relativity principle are important. However, in this treatise we do not touch such problems or do it merely exceptionally.

As is well known, the general fibre bundle (Q, M, π) does not need to be diffeomorphic with the Cartesian product $M \times Q^{\text{int}}$. And even if such a diffeomorphism does exist it is not in general unique in the sense that there is no canonically distinguished choice (there are exceptions like, e.g., tangent and cotangent bundles over Lie groups, bundles of linear frames or co-frames over Lie groups, and so on). Therefore, in general, the total motion $\varrho : \mathbb{R} \to Q$ does not split in a well-defined way into translational and internal motion. More precisely, only translational motion is well defined as a projection

$$\varrho_{\text{tr}} := \pi \circ \varrho : \mathbb{R} \to M.$$

Similarly, generalized velocity $\dot{\varrho}(t) \in T_{\varrho(t)}Q$ does not split into translational and internal velocities. Only the first one is well defined as a $T\pi$-projection of $\dot{\varrho}(t)$ to the tangent space $T_{\pi(\varrho(t))}M$; obviously, it is identical with $\dot{\varrho}_{\text{tr}}(t) \in T_{\varrho_{\text{tr}}(t)}M$. Without additional geometric objects the time-rate of internal configuration is not well defined.

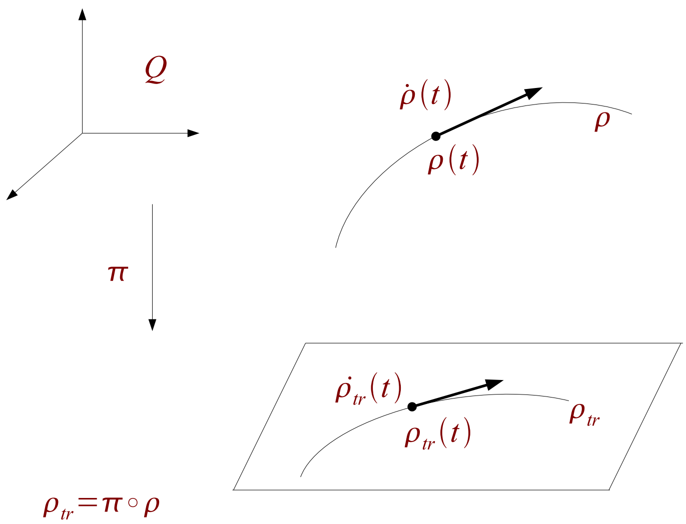

Fig. 1

The very concept of internal motion and internal trajectory is meaningful only when $Q = M \times Q^{\text{int}}$, i.e., when Q is the Cartesian product. In general, when the bundle (Q, M, π) is nontrivial (perhaps even not trivializable) and no additional connection-like structures in Q are defined, it is only the total motion in Q and the translational one in M that are well defined. The same concerns velocities.

In original Newton theory M is the Euclidean three-dimensional space. We shall consider the manifolds with a more general geometric structure and the fibre bundles (Q, M, π) over them. Having in view both the mathematical clarity and certain nonstandard physical applications we admit the general dimension, i.e., $\dim M = n$.

Chapter 1

Kinematics and canonical formalism for affine models

In a long series of earlier papers we have developed the theory of extended affinely-rigid bodies in Euclidean spaces [13, 14, 15, 16, 17, 18, 24, 25, 26, 27, 28, 31, 32, 33, 34, 36, 37, 38, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51]. Strictly speaking, kinematics of such objects preassumes only affine structure of the physical space. Affine constraints mean that all affine relations between material points remain invariant during admissible motions; material straight lines continue to be straight lines, their parallelism is preserved, etc. Assuming that all metrical relations are preserved one obtains the rigid body in the usual sense.

The concepts of extended metrically- and affinely-rigid bodies break down when the Euclidean or affine space is replaced by a differential manifold with geometry given by the metric tensor, affine connection, or both of them (interrelated or not). For example, let (M, g) be a Riemann space, where M is a manifold and g is a metric tensor defined on it. An extended continuous system of material points moving in M is metrically rigid if all infinitesimal distances are invariant during any admissible motion. Of course, then also finite along-geodetic distances are constant. If some standard reference configuration is fixed, the configuration space becomes identified with $\mathrm{Diff}(M, g)$, the isometry group, i.e., the group of all diffeomorphisms $\varphi : M \rightarrow M$ preserving the metric tensor, $\varphi^{*}g = g$. But it is well known that for the generic Riemann space (M, g) with not vanishing curvature tensor $R[g] \neq 0$ it is rather typical that the isometry group is trivial, i.e., $\mathrm{Diff}(M, g) = \{\mathrm{id}_M\}$. Obviously, the highest possible dimension of $\mathrm{Diff}(M, g)$ in an n-dimensional M equals $n(n+1)/2$. This highest dimension is attained only in constant-curvature spaces, like, e.g., spheres and pseudospheres in $\mathbb{R}^{n+1}$. And in fact, in such manifolds the concept of a finite extended rigid body continues to be well defined. But such manifolds, with $n(n+1)/2$ independent Killing vectors, are extremely exceptional in category of all Riemann manifolds. Let us also mention that even if $R[g] = 0$, i.e., (M, g) is locally Euclidean, but M is topologically not equivalent to $\mathbb{R}^n$ (e.g., torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ with the metric tensor inherited from $\mathbb{R}^n$), there may be obstacles against the existence of the global $n(n+1)/2$-dimensional isometry groups. So, in a generic Riemann manifold the concept of extended rigid body fails to be well defined.

The same concerns extended affinely-rigid bodies. Let (M, Γ) be an affine-connection space, i.e., a differential manifold M endowed with an affine connection Γ. This connection may be quite arbitrary, not necessarily one derived explicitly from some metric tensor g (at this stage the metric structure does not need to exist at all). Let ∇ denote the covariant differentiation corresponding to Γ. We say that $\varphi \in \mathrm{Diff}(M)$ is an affine transformation of (M, Γ), i.e., that $\varphi \in \mathrm{Diff}(M, \Gamma)$ if for any pair of vector fields X, Y on M the following holds:

$$\nabla_{\varphi_* X} (\varphi_* Y) = \varphi_* (\nabla_X Y),$$

where, as usual, ∇_X denotes the covariant differentiation along the vector field X. Roughly speaking, the above property means that the ∇-operation is "transparent" with respect to the transformation φ. In the usual flat affine space this is just an equivalent definition of the $n(n+1)$-dimensional affine group $\mathrm{GAf}(M)$. However, when the curvature tensor $\mathcal{R}[\Gamma]$ of the affine connection Γ is not vanishing, then, as a rule, the dimension of $\mathrm{Diff}(M, \Gamma)$ is smaller than $n(n+1)$ and the generic situation is that $\mathrm{Diff}(M, \Gamma) = \{\mathrm{id}_M\}$, i.e., the only affine transformation is the identity mapping. Therefore, in principle the concept of extended affinely-rigid body breaks down.

However, "very small" regions of (M, Γ) approximately look like flat affine spaces; similarly, "very small" regions of Riemannian manifolds (M, g) look like Euclidean spaces. Therefore, approximately, one can consider "very small" rigid and affinely-rigid bodies in non-Euclidean manifolds. This is always an approximation, the better one, the more body shrinks. Everything becomes rigorous in the limit of vanishing size. Roughly speaking, the body is not any longer injected in the physical space M, but in a tangent space $T_x M$, where $x \in M$ represents the spatial position of the body "as a whole" and is a remnant of the centre of mass position in the flat-space theory. And configurations of affine bodies in the linear space $T_x M$ may be identified with linear frames, i.e., ordered bases in $T_x M$.

This is a natural analogue of some description used in mechanics of extended affine bodies. Namely, let V be the linear space of translations in the physical affine space M. The configuration space of extended affinely-rigid body in M may be identified with $M \times F(V)$, where $F(V)$ denotes the manifold of linear frames in V. This is meant in the following sense: $x \in M$ is an instantaneous position of the centre of mass, and the frame $e = (e_1, \ldots, e_A, \ldots, e_n) \in F(V)$ is materially frozen into the body, i.e., co-moving with it. More precisely, if $a = (a^1, \ldots, a^n)$ are Lagrange (reference) coordinates of some material point (its identification labels), and configuration is given by $q = (x, e) = (x; e_1, \ldots, e_n) \in Q = M \times F(V)$, then the current spatial position $y \in M$ of this a-th point satisfies

$$\overrightarrow{xy} = a^K e_K,$$

where $\overrightarrow{xy}$ denotes the radius-vector of y with respect to the instantaneous position $x \in M$ of the centre of mass. Analytically,

$$y^i = x^i + e^i{}_K a^K$$

with respect to some Cartesian coordinates.

Essentially the same remains true for infinitesimal bodies injected in tangent spaces $T_x M$. They are somewhere placed in space, $x \in M$, and have the extra attached internal variables $e_K \in T_x M$, $K = \overline{1, n}$. Let us describe it in geometric terms.

The configuration space of infinitesimal affinely-rigid body moving in the physical space M is given by the manifold FM of linear frames in M,

$$Q = FM = \bigcup_{x \in M} F_x M,$$

where $F_x M$ denotes the manifold of linear frames in the tangent space $T_x M$. Obviously, just as $F(V)$ is an open submanifold of

$$V^n = \underbrace{V \times V \times \cdots \times V}_{n \text{ terms}},$$

so FM is an open submanifold in the Whitney sum of fibre bundles

$$\bigoplus_n TM = \bigcup_{x \in M} \underbrace{T_x M \times T_x M \times \cdots \times T_x M}_{n \text{ terms}}.$$

These open subsets consist of linearly independent linear n-tuples.

To be more precise, in the theory of the flat-space extended continuous bodies one should not use the total $F(V)$, but rather one of its connected components, e.g., one positively oriented with respect to some fixed standard orientation. And so is in mechanics of structured material points; by the way, it is tacitly assumed that M is orientable. Although, one can argue that if once the framework of classically imaginable extended bodies is left and affine degrees of freedom are essentially internal, the exotic orientation-changing motions might be perhaps admissible and even (at least mathematically) interesting. The problem becomes particularly interesting, perhaps just mathematically exciting, when M is not orientable, like, e.g., Möbius band.

The manifold FM carries a natural structure of the principal fibre bundle over the base M [19, 53]. The projection $\pi : FM \rightarrow M$ assigns to any linear frame the point at which it is attached, thus $\pi\left(F_x M\right) = x$. The structural group $\mathrm{GL}(n, \mathbb{R})$ acts on FM according to the standard rule; thus, for any $L \in \mathrm{GL}(n, \mathbb{R})$ we have that

$$FM \ni e = (\ldots, e_A, \ldots) \mapsto eL := (\ldots, e_B L^B{}_A, \ldots). \tag{1.1}$$

Obviously, $\dim FM = n(n+1)$; this is the number of degrees of freedom (n translational and n^2 internal ones). For any linear frame e there exists a unique dual co-frame $\widetilde{e} = (\ldots, e^A, \ldots)$, where

$$\left\langle e^A, e_B \right\rangle = \delta^A{}_B$$

and for any $x \in M$, $v \in T_x M$ and $p \in T^*_x M$ the symbol $\langle p, v\rangle$ denotes the evaluation of the linear function p on the vector v.

The manifold of all co-frames, i.e.,

$$Q^* = F^*M = \bigcup_{x \in M} F^*_x M,$$

is canonically diffeomorphic with FM just in the sense of duality. Therefore, the configuration space of infinitesimal affine body may be represented either as FM or F^*M; it depends on the particular problem which representation is more convenient. Just as previously, F^*M is an open subset of the Whitney sum

$$\bigoplus_n T^*M = \bigcup_{x \in M} \underbrace{T^*_x M \times T^*_x M \times \cdots \times T^*_x M}_{n \text{ terms}};$$

it consists of linearly independent n-tuples. The natural projection onto the base manifold will be denoted by $\pi^* : F^*M \rightarrow M$, and then $\pi^*\left(F^*_x M\right) = x$. Obviously, F^*M also carries the structure of the principal fibre bundle with $\mathrm{GL}(n, \mathbb{R})$ as a structural group:

$$F^*M \ni \widetilde{e} = \left(\ldots, e^A, \ldots\right) \mapsto \widetilde{e}L := \left(\ldots, L^{-1A}{}_B e^B, \ldots\right) \tag{1.2}$$

for any $L \in \mathrm{GL}(n, \mathbb{R})$.

There is one subtle point, namely, $\mathrm{GL}(n,\mathbb{R})$ is not connected. It consists of two connected components, i.e., the subgroup $\mathrm{GL}^+(n,\mathbb{R})$ of positive-determinant matrices and $\mathrm{GL}^-(n,\mathbb{R})$ is the coset (not a subgroup, of course) of negative-determinant matrices. In theory of Lie groups, fibre bundles and connections one usually deals with connected groups. Thus, it seems rather natural to deal with reductions to the subgroup $\mathrm{GL}^+(n,\mathbb{R})$. But this has again to do with orientability of M. Then the unconnected manifold FM is replaced by F^+M, i.e., the manifold of linear frames positively oriented with respect to some fixed orientation on M.

Any system of local coordinates x^i, $i=\overline{1,n}$, on M induces local coordinates $(x^i, e^i{}_A)$, $A=\overline{1,n}$, on FM and $(x^i, e^A{}_i)$ on F^*M, where $e^i{}_A$, $e^A{}_i$ are respectively components of e_A, e^A with respect to coordinates x^i, and

$$e^A{}_i e^i{}_B = \delta^A{}_B, \qquad e^i{}_A e^A{}_j = \delta^i{}_j.$$

To avoid the crowd of symbols we do not distinguish graphically between x^i and their pull-backs to FM and F^*M.

In certain problems it is convenient to admit the local, x-dependent action of the structural group, like in gauge theories. This means that we consider (sufficiently smooth) fields $L : M \to \mathrm{GL}(n,\mathbb{R})$; they form an infinite-dimensional group under the pointwise multiplication,

$$(L_1L_2)(x) = L_1(x)L_2(x).$$

And this group acts, also in a pointwise way, on FM and F^*M. Thus, if $e \in F_xM$, $\widetilde{e} \in F^*_xM$, then the action of L is given by

$$e \mapsto eL(x), \qquad \widetilde{e} \mapsto \widetilde{e}L(x). \tag{1.3}$$

From the mechanical point of view the structural action of $\mathrm{GL}(n,\mathbb{R})$ on $Q = FM$ and $Q^* = F^*M$ corresponds to material transformations. We shall use the term "micromaterial transformations". This is exactly the infinitesimal limit of the usual material transformations. In fact, material points with affine internal degrees of freedom may be interpreted as affinely-rigid bodies injected into instantaneous tangent spaces T_xM, where $x \in M$ is the spatial position of the body. But it is obvious that $e \in F_xM$ is canonically identical with some linear isomorphism of $\mathbb{R}^n$ onto T_xM (similarly, $\widetilde{e} \in F^*_xM$ is a linear isomorphism of T_xM onto $\mathbb{R}^n$). In this way, $\mathbb{R}^n$ plays the role of the "micromaterial space" (Lagrange variables) and T_xM is the "microphysical space" (Euler variables) of infinitesimal affinely-rigid body. The frame $e \in F_xM$, i.e., the "placement" is then exactly the counterpart of what was denoted by $\varphi \in \mathrm{LI}(U,V)$ in our papers about extended flat-space bodies [31, 32, 33, 34, 36, 37, 38, 41, 42, 44, 45, 46, 47, 48, 49, 50, 51].

And what does correspond to spatial or physical transformations, i.e., what are "microspatial" kinematical symmetries? Here the situation is more complicated. Obviously, microspatial transformations must act on the left in tangent spaces $T_x M$. But for different points $x \in M$ the tangent spaces $T_x M$ are logically different linear spaces and their linear groups $\mathrm{GL}(T_x M)$ are also logically different sets. Without additional strong geometric structures there is no natural isomorphism between different $\mathrm{GL}(T_x M)$. So, there is only one possibility, in an essential way infinite-dimensional. Namely, the fields T of mixed non-degenerate second-order tensors on M,

$$M \ni x \mapsto T_x \in \mathrm{GL}(T_x M) \subset \mathrm{L}(T_x M) \simeq T^1_1(T_x M),$$

give rise to the following transformations of FM, F^*M:

$$F_x M \ni e = (\ldots, e_A, \ldots) \quad \mapsto \quad (\ldots, T_x \circ e_A, \ldots), \tag{1.4}$$

$$F^*_x M \ni \widetilde{e} = (\ldots, e^A, \ldots) \quad \mapsto \quad \left(\ldots, e^A \circ T_x^{-1}, \ldots\right). \tag{1.5}$$

In a structure-less manifold M and in a connection manifold (M, Γ) with not vanishing curvature tensor $\mathcal{R}(\Gamma)$, there are no natural isomorphisms between different $T_x M$, therefore, nothing like the "constancy" of T may be defined. If the components field $T^i{}_j$ is accidentally constant in some coordinates x^i, then in other coordinates it is no longer true. Therefore, by its very nature the above group of left-acting transformations is infinite-dimensional, flexible. Of course, in the right-hand side action of $\mathrm{GL}(n, \mathbb{R})$ (1.2) L may be put x-dependent like in gauge theories of fields and continua, but need not to be so; its constancy is well defined. $L \in \mathrm{GL}(n, \mathbb{R})$ are just matrices by their very nature, not matrix representants of linear mappings in $T_x M$ with respect to some coordinates.

When some coordinates x^i are fixed in M, the second-order mixed tensor field T is represented by a system of functions, i.e., components $T^i{}_j(x^a)$,

$$T = T^i{}_j(x) \frac{\partial}{\partial x^i} \otimes dx^j.$$

Analytically, the action of T on the configuration space Q is described by

$$\left(\ldots, x^a, \ldots; \ldots, e^i{}_A, \ldots\right) \mapsto \left(\ldots, x^a, \ldots; \ldots, T^i{}_j(x) e^j{}_A, \ldots\right). \tag{1.6}$$

Similarly, x-dependent $L \in \mathrm{GL}(n, \mathbb{R})$ act as follows:

$$\left(\ldots, x^a, \ldots; \ldots, e^i{}_A, \ldots\right) \mapsto \left(\ldots, x^a, \ldots; \ldots, e^i{}_B L^B{}_A(x), \ldots\right). \tag{1.7}$$

In particular, this formula holds for global, x-independent $L \in \mathrm{GL}(n, \mathbb{R})$. It is seen that when $x \in M$ is kept fixed, these are just the well-known formulas for an affinely-rigid body in a flat space $T_x M$. The difference is that there is no counterpart of translations and general affine mappings in M and in the total FM. The reason is that, as mentioned, even if M is endowed with some affine connection Γ with not vanishing curvature tensor, the group of affine transformations is generically trivial or its dimension is smaller than $n(n+1)$. And if M is completely amorphous, i.e., even if any affine connection is not fixed, then it is only the total diffeomorphism group $\mathrm{Diff}(M)$ that may replace the group of spatial affine transformations in mechanics of extended affine bodies in a flat space.

In mechanics of affine extended bodies [31, 32, 33, 34, 36, 37, 38, 41, 42, 44, 45, 46, 47, 48, 49, 50, 51] we considered various kinds of additional constraints; for obvious reasons the most important of them were metrical constraints, i.e., rigid body in the literal sense. Not only affine relations between material points are then preserved but also metrical ones, i.e., distances and angles. In flat-space theory one is dealing then, also on the kinematical level, with the Euclidean structure (M, V, g), where V is the linear space of translations in M, and $g \in V^* \otimes V^*$ is the metric tensor of M. The configuration space of the rigid body may be identified with $Q = M \times F(V, g)$, where $F(V, g) \subset F(V)$ is the manifold of g-orthonormal frames $e = (e_1, \ldots, e_A, \ldots, e_n)$, i.e.,

$$g(e_A, e_B) = g_{ij} e^i{}_A e^j{}_B = \delta_{AB}.$$

More precisely, one should use rather some connected component of $F(V, g)$, e.g., the manifold $F^+(V, g)$ of g-orthonormal frames positively oriented with respect to some fixed orientation of M.

When (M, g) is a general Riemann manifold, then, as a rule, there are no extended rigid bodies with the usual number of $n(n+1)/2$ degrees of freedom; the only exception are constant-curvature spaces. In a generic case the not trivial isometries do not exist at all. But just as in affine theory we can speak about infinitesimal rigid bodies. They describe in a good approximation the behaviour of very small "almost rigid bodies", in which the changes of distances between neighbouring particles are higher-order small in comparison with the initial distances themselves.

The configuration space of infinitesimal rigid body in (M, g) may be identified with $F(M, g)$, i.e., the manifold of all g-orthonormal frames in all tangent spaces of M. Again, to be more precise, if M is orientable, we should restrict ourselves to $F^+(M, g)$, i.e., the connected manifold of g-orthonormal frames positively oriented with respect to some fixed orientation in M. Obviously,

$F(M, g)$, $F^{+}(M, g)$ are $n(n+1)/2$-dimensional manifolds; there are n translational degrees of freedom (M) and $n(n-1)/2$ rotational ones (fibres $(F_x M, g_x)$). Just as in affine model, it does not matter whether we use $F(M, g)$ or the manifold $F^{*}(M, g)$ of g-orthonormal co-frames. Projections onto the base manifold M are just the restrictions of the previous π, π^{*} to submanifolds $F(M, g)$, $F^{*}(M, g)$; for brevity we denote them by the same symbols. The structure groups of $F(M, g)$, $F^{+}(M, g)$ are respectively $\mathrm{O}(n, \mathbb{R}) \subset \mathrm{GL}(n, \mathbb{R})$, $\mathrm{SO}(n, \mathbb{R}) \subset \mathrm{GL}(n, \mathbb{R})$; they act on the bundle manifolds on the right, just in the sense of formulas (1.1), (1.2). Similarly, the left-hand-side spatial transformations are given by (1.4) and (1.5), where now the field T takes values in orthogonal groups of $(T_x M, g_x)$, i.e.,

$$\begin{aligned} M \ni x &\mapsto T_x \in \mathrm{O}\left(T_x M, g_x\right) \subset \mathrm{GL}(T_x M), \\ M \ni x &\mapsto T_x \in \mathrm{SO}\left(T_x M, g_x\right) \subset \mathrm{O}\left(T_x M, g_x\right); \end{aligned}$$

obviously, $g_x \in T_x^{*}M \otimes T_x^{*}M$ is the metric tensor of $T_x M$, i.e., the value of the field g at $x \in M$.

What concerns the very description of degrees of freedom, infinitesimal affinely-rigid body is well defined in any quite amorphous differential manifold. Obviously, the usual, i.e., metrically-rigid body is meaningful only when M is endowed with some metric tensor g.

But, as mentioned, even on the purely kinematical level there are some difficulties with systems with internal degrees of freedom. Namely, for any motion $\varrho : \mathbb{R} \to FM$ it is only the tangent vector $\dot{\varrho}(t) \in T_{\varrho(t)}FM$ and its projection $\dot{\varrho}_{\mathrm{tr}}(t) \in T_{\pi(\varrho(t))}M$ (see Fig. 2) that are well defined velocities, i.e., respectively the total generalized and translational velocities.

There is no well-defined time rate of internal variables evolution. The minimal geometric structure necessary to define it is a connection on the fibre bundle (Q, M, π). In mechanics of infinitesimal affine bodies and infinitesimal gyroscopes this is simply the affine connection. Therefore, from now on we assume that some affine (more precisely linear) connection Γ is fixed in M. Analytically it is given by the system of components $\Gamma^{i}{}_{jk}$ transforming under the change of coordinates x^{i} on M according to the well-known linear-inhomogeneous rule. In modern differential geometry [19, 53] linear connection is described by some differential one-form ω on FM (or $F(M, g)$) taking values in the Lie algebra of the structural group $\mathrm{GL}(n, \mathbb{R})' \simeq \mathrm{L}(n, \mathbb{R})$ (or $\mathrm{SO}(n, \mathbb{R})'$, i.e., the space of skew-symmetric matrices). It satisfies certain conditions that are quoted in [19, 53]. Analytically ω is represented by the system of differential one-forms $\omega^{K}{}_{L}$ related in the following way to coordinate-dependent quantities $\Gamma^{i}{}_{jk}$:

$$\omega^{A}{}_{B} = e^{A}{}_{i}\left(de^{i}{}_{B} + \Gamma^{i}{}_{jk} e^{j}{}_{B} dx^{k}\right).$$

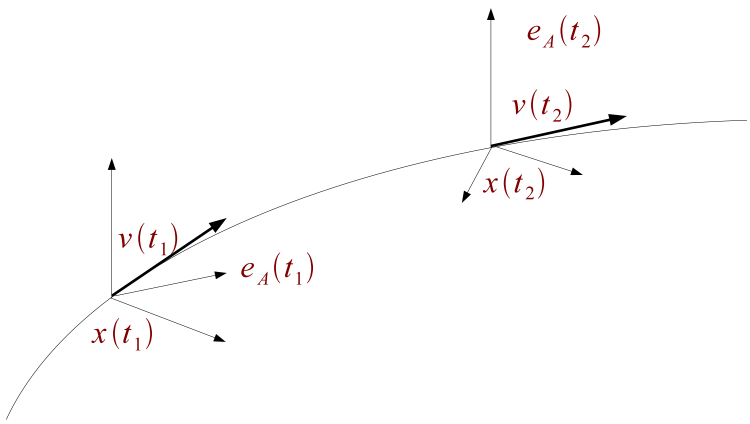

Fig. 2

One uses also the connection-independent canonical $\mathbb{R}^n$-valued differential one-form θ on FM; it is intrinsically defined and its coordinate representation is given by

$$\theta^A = e^A{}_i dx^i.$$

It vanishes when evaluated on vectors tangent to the fibres F_xM, $F_x(M,g)$, i.e., vertical ones. The kernel of ω_e is transversal to the space of vertical vectors; its elements are by definition "horizontal" vectors in T_eFM, $T_eF(M,g)$. The subspaces of vertical and horizontal vectors at e are denoted respectively by V_e, H_e. Obviously, $T_eFM = V_e \oplus H_e$, and similarly for $F(M,g)$. Roughly speaking, horizontal vectors establish isomorphisms between fibres over infinitesimally remote points of M. Therefore, infinitesimally, the concepts like the change of internal state and the velocity of internal motion become meaningful.

At any e, the co-vectors $\omega_e, \theta_e \in T^*_eFM$ ($T^*_eF(M,g)$ in the gyroscopic case) form a basis of T^*_eFM. The dual basis in T_eFM ($T_eF(M,g)$) consists of vectors denoted by $E^A{}_B$, H_A and

$$\langle\omega^K{}_L, E^A{}_B\rangle = \delta^K{}_B\delta^A{}_L, \qquad \langle\omega^K{}_L, H_A\rangle = 0,$$
$$\langle\theta^K, E^A{}_B\rangle = 0, \qquad \langle\theta^K, H_A\rangle = \delta^K{}_A.$$

One can easily show that, after identifying vector fields with differential operators [19, 53], we have that

$$E^K{}_L = e^i{}_L \frac{\partial}{\partial e^i{}_K}, \qquad H_L = e^i{}_L\left(\frac{\partial}{\partial x^i} - \Gamma^k{}_{ji}e^j{}_A\frac{\partial}{\partial e^k{}_A}\right).$$

At any point e, the corresponding vectors $\left(E^K{}_L\right)_e$ are vertical, i.e., tangent to the fibres, $\left(E^K{}_L\right)_e \in V_e$, whereas $(H_K)_e$ are horizontal, $(H_K)_e \in H_e$. Dually to the situation with co-vectors $\omega^A{}_B$, θ^A, now $E^K{}_L$ are connection-independent, whereas H_L depend explicitly on the connection. In the literature [19, 53] H_K are referred to as standard horizontal vector fields and $E^K{}_L$ as fundamental vector fields. $E^K{}_L$ are infinitesimal generators of the action of the structural group $\mathrm{GL}(n,\mathbb{R})$ on FM. Roughly speaking, H_K generate the parallel transport in the sense of Γ.

The splitting $T_eFM = V_e \oplus H_e$ enables one to decompose every vector at e into vertical and horizontal parts. In particular, this may be done for generalized velocities $\dot{\varrho}(t) \in T_{\varrho(t)}FM$. Projecting the horizontal component to M, we do not obtain anything new, just the translational velocity $\dot{\varrho}_{\rm tr}(t) \in T_{\pi(\varrho(t))}M$. The vertical part, however, is a new, Γ-depending object. It is a measure of the internal velocity, a kind of the time rate of internal configuration. And, as expected, it simply coincides with the system of covariant derivatives of the frame vectors e_A along the curve describing translational motion. If x^i, $e^i{}_A$ are coordinates on FM induced by those x^i on M, and if instead of sophisticated symbols $x^i \circ \varrho$, $e^i{}_A \circ \varrho$ we simply write $x^i(t)$, $e^i{}_A(t)$ for the time dependence of generalized coordinates of the object, then translational velocity v has the components

$$v^i = \frac{dx^i}{dt},$$

whereas the internal velocity is given as follows:

$$V^i{}_A = \frac{De^i{}_A}{Dt} = \frac{de^i{}_A}{dt} + \Gamma^i{}_{jk}\left(x(t)\right) e^j{}_A \frac{dx^k}{dt}.$$

It is very important to stress that if (M,Γ) is non-Euclidean, i.e., the curvature and torsion tensors do not vanish:

$$\begin{aligned} \mathcal{R}^i{}_{jkl} &= \partial_k \Gamma^i{}_{jl} - \partial_l \Gamma^i{}_{jk} + \Gamma^i{}_{ak}\Gamma^a{}_{jl} - \Gamma^i{}_{al}\Gamma^a{}_{jk} \neq 0, \\ S^i{}_{jk} &= \frac{1}{2}\left(\Gamma^i{}_{jk} - \Gamma^i{}_{kj}\right) \neq 0, \end{aligned}$$

then the system $\left(v^i, V^i{}_A\right)$ is the aholonomic velocity in the sense that there are no coordinates q^i, $q^i{}_A$ in FM for which the following could hold:

$$v^i = \frac{dq^i}{dt}, \qquad V^i{}_A = \frac{dq^i{}_A}{dt}.$$

Generalized velocities at $e \in \pi^{-1}(x)$ are represented by the $(n+1)$-tuples of vectors $(V; \ldots, V_A, \ldots) \in (T_xM)^{n+1}$ attached at $x = \pi(e) \in M$. Therefore, the affine connection Γ enables one to identify the tangent bundle TFM with the following fibre bundle over M:

$$\mathcal{T}FM = \bigcup_{x\in M} F_xM \oplus (T_xM)^{n+1} \subset \bigcup_{x\in M} (T_xM)^{2n+1}.$$

It is an open subset of the last Whitney sum; namely, the first n-tuple is the submanifold of $(T_xM)^n$ consisting of linearly independent systems. Obviously, for any fixed affine connection Γ the above-mentioned diffeomorphism of TFM onto $\mathcal{T}FM$ is canonical.

Similarly, canonical momenta at $e \in \pi^{-1}(x)$ are represented by the $(n+1)$-tuples of co-vectors at $x = \pi(e)$, i.e., $\left(P; \ldots, P^A, \ldots\right) \in (T_x^*M)^{n+1}$.

It is seen again that Γ establishes a distinguished diffeomorphism of the cotangent bundle T^*FM (phase space of the system) with the bundle

$$\mathcal{T}^*FM = \bigcup_{x\in M} F_xM \oplus (T_x^*M)^{n+1} \subset \bigcup_{x\in M} (T_xM)^n \oplus (T_x^*M)^{n+1}.$$

Denoting generalized velocities by

$$v^i = \frac{dx^i}{dt}, \qquad v^i{}_A = \frac{de^i{}_A}{dt}$$

and their conjugate canonical momenta by p_i, $p^A{}_i$, we have that

$$p_iv^i + p^A{}_iv^i{}_A = P_iV^i + P^A{}_iV^i{}_A,$$

and

$$V^i = v^i, \qquad V^i{}_A = v^i{}_A + \Gamma^i{}_{jk}(x)e^j{}_Av^k,$$
$$P_i = p_i - e^j{}_Ap^A{}_k\Gamma^k{}_{ji}, \qquad P^A{}_i = p^A{}_i.$$

Let us observe the following anti-dualism: it is easily seen that V^i and $P^A{}_i$ are connection-independent, whereas P_i and $V^i{}_A$ depend explicitly on Γ. Geometric reasons for this are that that V is simply the π-projection of generalized velocity $\dot{\varrho}(t)$ to M and $P^A{}_i$ are components of the linear functional on $T_{\varrho(t)}F_{x(t)}M$ $(x(t) = \pi(\varrho(t)))$ which is simply the usual restriction of some functional on $T_{\varrho(t)}FM$ to the subspace $T_{\varrho(t)}F_{x(t)}M$. The restriction procedure is obviously connection-independent.

Remark: $v^i{}_A$ are not components of vectors in T_xM, and similarly, p_i are not elements of covectors. Only v^i and $p^A{}_i$ are so. The total systems $\left(v^i, v^i{}_A\right)$, $\left(p_i, p^A{}_i\right)$ represent the higher-floor vectors and covectors in T_eFM. The connection-dependent systems $V^i{}_A$, P_i are respectively components of vectors and covectors in T_xM (elements of T_xM itself and T_x^*M).

Quite a similar reasoning works for infinitesimal gyroscope, i.e., for the bundle $F(M,g)$. The situation there is more complicated technically, because the quantities $e^{i}{}_{A}$ are no longer independent, i.e., they satisfy the orthonormality conditions

$$g_{ij}e^{i}{}_{A}e^{j}{}_{B} = \delta_{AB}.$$

This fact creates some problems in equations of motion; to overcome them one uses an auxiliary technical tool, namely, a field of linear orthonormal frames defined all over the manifold M. We return to this question later on.

In mechanics of extended affinely- and metrically-rigid bodies one uses the concepts of affine velocity (Eringen's "gyration") and affine spin (affine momentum, hypermomentum). They may be as well defined for infinitesimal objects, i.e., for internal degrees of freedom.

Every tensor object in the tangent space T_eFM or $T_eF(M,g)$ may be expressed in terms of its components with respect to the frame e itself. In particular, this concerns the translational velocity V and the system of internal velocities $(\ldots, V_A, \ldots)$. Similarly, the covariant conjugate momenta P, $\big(\ldots, P^{A}, \ldots\big)$ may be expanded with respect to the dual co-frame $\widetilde{e}$. So, we have that

$$V = \widehat{V}^{A}e_{A}, \qquad V_{A} = e_{B}\widehat{\Omega}^{B}{}_{A},$$

i.e.,

$$\widehat{V}^{A} = \langle e^{A}, V\rangle = e^{A}{}_{i}V^{i}, \qquad \widehat{\Omega}^{A}{}_{B} = \langle e^{A}, V_{B}\rangle = e^{A}{}_{i}V^{i}{}_{B},$$

or explicitly, using differentiation symbols,

$$\widehat{\Omega}^{A}{}_{B} = \left\langle e^{A}, \frac{De_{B}}{Dt}\right\rangle = e^{A}{}_{i}\frac{De^{i}{}_{B}}{Dt}.$$

This expression is exactly the co-moving affine velocity known from the theory of extended affine bodies in a flat space. $\widehat{V}^{A}$ are co-moving components of the translational velocity. Geometrically, these objects belong to the matrix spaces, i.e.,

$$\widehat{\Omega} \in \mathrm{L}(n,\mathbb{R}) \simeq \mathrm{GL}(n,\mathbb{R})', \qquad \widehat{V} \in \mathbb{R}^{n} \simeq \mathrm{L}(1,n;\mathbb{R}).$$

But having e instantaneously fixed, we can associate with these numerical objects the corresponding quantities in the tangent space T_xM, where $x = \pi(e)$ $(e \in F_xM)$. For $\widehat{V}$ this is just the usual translational velocity $V \in T_xM$:

$$V^{i} = \frac{dx^{i}}{dt}.$$

The quantity $\widehat{\Omega}$ and the frame e give rise to the object

$$\Omega = \widehat{\Omega}^{A}{}_{B}e_{A} \otimes e^{B}, \qquad \Omega^{i}{}_{j} = V^{i}{}_{A}e^{A}{}_{j},$$

i.e., explicitly in terms of derivatives:

$$\Omega = \frac{De_A}{Dt} \otimes e^A, \qquad \Omega^i{}_j = \frac{De^i{}_A}{Dt} e^A{}_j = e^i{}_A \widehat{\Omega}^A{}_B e^B{}_j.$$

Geometrically, they are linear transformations in tangent spaces, i.e.,

$$\Omega \in \mathrm{L}(T_x M) \simeq T_x M \otimes T_x^* M = \mathrm{GL}(T_x M)',$$

where, obviously, $x = \pi(e)$ ($e \in F_x M$).

Just as in mechanics of extended affine bodies, $\Omega^i{}_j$ are spatial (laboratory) components of the affine velocity, i.e., they represent what Eringen used to call "gyration" [10, 11]. Geometrically it is important that $\widehat{\Omega}$, Ω belong respectively to Lie algebras of the groups $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(T_x M)$.

The same may be done for the covariant canonical momenta P and P^A:

$$P = \widehat{P}_A e^A, \qquad P^A = \widehat{\Sigma}^A{}_B e^B,$$

i.e.,

$$\widehat{P}_A = \langle P, e_A \rangle = P_i e^i{}_A, \qquad \widehat{\Sigma}^A{}_B = \langle P^A, e_B \rangle = P^A{}_i e^i{}_B.$$

Just as previously, $\widehat{P} \in \mathbb{R}^n$, $\widehat{\Sigma} \in \mathrm{L}(n, \mathbb{R})$, but strictly speaking one means here $\mathbb{R}^n$ as identified in a Cartesian way with its own dual $\mathbb{R}^{n*}$; similarly, $\mathrm{L}(n, \mathbb{R})$ plays here a role of the Lie co-algebra $\mathrm{L}(n, \mathbb{R})^*$ of $\mathrm{GL}(n, \mathbb{R})$. One does not notice this subtle distinction because $\mathrm{L}(n, \mathbb{R})^*$ and $\mathrm{L}(n, \mathbb{R})$ are canonically identified via the pairing

$$\langle A, B \rangle = \mathrm{Tr}(AB).$$

In analogy to mechanics of extended bodies in flat spaces we say that the matrix elements of $\widehat{\Sigma}$ are co-moving components of the affine spin (hypermomentum). Just as in flat-space theory, the laboratory (spatial) description is based on the quantity

$$\Sigma = \widehat{\Sigma}^A{}_B e_A \otimes e^B, \qquad \Sigma^i{}_j = e^i{}_A P^A{}_j = e^i{}_A \widehat{\Sigma}^A{}_B e^B{}_j.$$

So, $\widehat{\Sigma}^A{}_B$ are Hamiltonian generators of the structural group $\mathrm{GL}(n, \mathbb{R})$ (micromaterial global transformations) (1.1), (1.7). And similarly, $\Sigma^i{}_j$ are Hamiltonian generators of the local group of microspatial transformations (1.4), (1.5), (1.6).

If the configuration $e \in FM$ is fixed, then the velocities V^i, $V^i{}_A$ contain exactly the same information as V^i, $\Omega^i{}_j$ or $\widehat{V}^A$, $\widehat{\Omega}^A{}_B$. Similarly, the momenta P_i, $P^A{}_i$ are equivalent to P_i, $\Sigma^i{}_j$ and $\widehat{P}_A$, $\widehat{\Sigma}^A{}_B$. Using more sophisticated terms we would say that the state space $\mathcal{T}FM$ is naturally diffeomorphic with the manifolds

$$\begin{aligned}\mathcal{T}_{\Omega}FM &:= \bigcup_{x\in M} F_xM \oplus T_xM \oplus \mathrm{L}(T_xM),\\ \mathcal{T}_{\widehat{\Omega}}FM &:= \bigcup_{x\in M} F_xM \oplus \mathbb{R}^n \oplus \mathrm{L}(n,\mathbb{R}).\end{aligned}$$

Similarly, the phase space $\mathcal{T}^*FM$ is canonically diffeomorphic with

$$\begin{aligned}\mathcal{T}^*_{\Sigma}FM &:= \bigcup_{x\in M} F_xM \oplus T^*_xM \oplus \mathrm{L}(T_xM),\\ \mathcal{T}^*_{\widehat{\Sigma}}FM &:= \bigcup_{x\in M} F_xM \oplus \mathbb{R}^n \oplus \mathrm{L}(n,\mathbb{R}).\end{aligned}$$

Let us notice that there is a subtle distinction between $\mathcal{T}_{\widehat{\Omega}}FM$ and $\mathcal{T}^*_{\widehat{\Sigma}}FM$ indistinguishable in our simplified notation. Namely, in the latter space the $\mathbb{R}^n$-term consists of row numerical vectors, i.e., linear functionals on the space of numerical column vectors (so, strictly speaking, one deals with $\mathbb{R}^{n*}$). And $\mathrm{L}(n,\mathbb{R})$ is canonically isomorphic with $\mathrm{L}(n,\mathbb{R})^*$. (**Remark:** this isomorphism is canonical for $\mathrm{L}(V)$, $\mathrm{L}(V)^*$, where V is an arbitrary linear space. This is no longer the case for V, V^* themselves.) Obviously, the mentioned identifications between state manifolds may be interpreted as natural ones only on the basis of some fixed affine connection Γ on M.

The cotangent bundle $P = T^*FM$ carries a natural symplectic structure [1, 43, 53] and is used as the mechanical phase space of our problem. As just mentioned, the affine connection Γ fixes some diffeomorphism of P onto the manifold $\mathcal{P} = \mathcal{T}^*FM$. This diffeomorphism enables one to carry over to $\mathcal{P}$ the intrinsic symplectic geometry of $P = T^*FM$. But there is one delicate point which, when overlooked, may lead to serious mistakes. Namely, unlike the previously used coordinates in T^*FM, the quantities x^i, $e^i{}_A$, P_i, $P^A{}_i$ fail to be canonical (Darboux) coordinates for the symplectic structure Γ-transferred to $\mathcal{T}^*FM$. After some essentially easy although sometimes technically embarrassing calculations one obtains the system of basic Poisson brackets. So, just like in mechanics of extended affine bodies, we have the obvious rules

$$\begin{aligned}&\{x^i, x^j\} = 0, \quad \{e^i{}_A, e^j{}_B\} = 0, \quad \{x^i, e^j{}_A\} = 0, \quad \{P^A{}_i, P^B{}_j\} = 0,\\ &\{P^A{}_i, x^j\} = 0, \qquad \{e^i{}_A, P^B{}_j\} = \delta^i{}_j\delta^B{}_A, \qquad \{x^i, P_j\} = \delta^i{}_j.\end{aligned}$$

However, further on one obtains more complicated expressions explicitly dependent on the Γ-geometry of M:

$$\{P_i, P_j\} = \Sigma^k{}_l\mathcal{R}^l{}_{kij}, \quad \{P_i, P^A{}_j\} = -P^A{}_k\Gamma^k{}_{ji}, \quad \{P_i, e^j{}_A\} = e^k{}_A\Gamma^j{}_{ki},$$

where, obviously, $\mathcal{R}$ denotes the curvature tensor of Γ; we use the convention

$$\mathcal{R}^{a}{}_{bij} = \Gamma^{a}{}_{bj,i} - \Gamma^{a}{}_{bi,j} + \Gamma^{a}{}_{ci}\Gamma^{c}{}_{bj} - \Gamma^{a}{}_{cj}\Gamma^{c}{}_{bi}$$

(comma, as usual in the tensor calculus, denotes the partial differentiation with respect to the indicated coordinate). The latter three Poisson brackets mean, roughly speaking, that P_i are Hamiltonian generators of parallel transports of our state variables. One can also show that

$$\{\Sigma^{i}{}_{j}, \Sigma^{k}{}_{l}\} = \delta^{i}{}_{l}\Sigma^{k}{}_{j} - \delta^{k}{}_{j}\Sigma^{i}{}_{l}.$$

In these formulas we recognize structure constants of the linear group. As expected, this means that $\Sigma^{i}{}_{j}$ are basic Hamiltonian generators of (1.4), (1.5), i.e., roughly speaking, of the system of groups $\mathrm{GL}(T_xM)$. It is easy to obtain the relationship

$$\{P_i, \Sigma^{k}{}_{j}\} = \Sigma^{l}{}_{j}\Gamma^{k}{}_{li} - \Sigma^{k}{}_{l}\Gamma^{l}{}_{ji},$$

that is also compatible with the mentioned interpretation of P_i as Hamiltonian generators of parallel transports.

If some function F depends only on the configuration, i.e., on the FM-variables (but not on P_i and $P^{A}{}_{i}$), then

$$\{\Sigma^{i}{}_{j}, F\} = -E^{i}{}_{j}F = -e^{i}{}_{A}\frac{\partial F}{\partial e^{j}{}_{A}},$$

where

$$E^{i}{}_{j} = e^{i}{}_{K}e^{L}{}_{j}E^{K}{}_{L} = e^{i}{}_{K}\frac{\partial}{\partial e^{j}{}_{K}}.$$

The Poisson brackets involving co-moving components are as follows:

$$\{\widehat{P}_A, \widehat{P}_B\} = \widehat{\Sigma}^{K}{}_{L}\mathcal{R}^{L}{}_{KAB} - 2\widehat{P}_K S^{K}{}_{AB}, \qquad \{\widehat{\Sigma}^{A}{}_{B}, \widehat{P}_C\} = -\widehat{P}_B\delta^{A}{}_{C},$$
$$\{\widehat{\Sigma}^{A}{}_{B}, \widehat{\Sigma}^{C}{}_{D}\} = \delta^{C}{}_{B}\widehat{\Sigma}^{A}{}_{D} - \delta^{A}{}_{D}\widehat{\Sigma}^{C}{}_{B}, \qquad \{\Sigma^{i}{}_{j}, \widehat{\Sigma}^{A}{}_{B}\} = 0,$$

where $\mathcal{R}^{L}{}_{KAB}$, $S^{K}{}_{AB}$ are respectively co-moving components of the curvature and torsion tensors of Γ with respect to the instantaneous internal configuration e, i.e.,

$$\begin{aligned}\mathcal{R}^{L}{}_{KAB} &:= e^{L}{}_{i}\mathcal{R}^{i}{}_{jmn}e^{j}{}_{K}e^{m}{}_{A}e^{n}{}_{B},\\ S^{K}{}_{AB} &:= e^{K}{}_{i}S^{i}{}_{jm}e^{j}{}_{A}e^{m}{}_{B}, \qquad S^{i}{}_{jm} = \frac{1}{2}\left(\Gamma^{i}{}_{jm} - \Gamma^{i}{}_{mj}\right).\end{aligned}$$

Let us observe a characteristic difference between the spatial and co-moving representations of Poisson brackets. Namely, the latter ones are “almost” identical with the basic commutation relations (structure constants) for the affine group $\mathrm{GAf}(n,\mathbb{R})$. The "almost" concerns the brackets $\{\widehat{P}_A, \widehat{P}_B\}$ which do not vanish if the connection Γ is not completely flat (in the sense that both the curvature and torsion tensors vanish). The co-moving brackets enable one to interpret $\widehat{P}_A$ as a Hamiltonian generator of parallel transports along the A-th legs of the frames e.

Let us also note certain additional and convenient Poisson brackets. If F depends only on the configuration, then

$$\begin{aligned}\{\widehat{\Sigma}^A{}_B, F\} &= -E^A{}_B F = -e^i{}_B \frac{\partial F}{\partial e^i{}_A},\\ \{\widehat{P}_A, F\} &= -H_A F, \qquad \{P_i, F\} = -H_i F,\end{aligned}$$

where

$$H_i = e^A{}_i H_A = \frac{\partial}{\partial x^i} - \Gamma^k{}_{ji} e^j{}_B \frac{\partial}{\partial e^k{}_B}.$$

Again we conclude that P_i are Hamiltonian generators of parallel transports along the i-th coordinate axes, and $\widehat{P}_A$ are generators of parallel transports along the A-th legs of the frames e.

In the theory of an extended affinely-rigid body in Euclidean or affine space the quantities P_i, $P^A{}_i$, $\Sigma^i{}_j$, $\widehat{\Sigma}^A{}_B$ have a very natural geometric interpretation based on some transformation groups acting in the configuration space. Let us remind that in the flat affine case the configuration space FM trivializes to the Cartesian product $M \times F(V)$, where $F(V)$ is the manifold of frames in V. The most important transformation groups acting in $M \times F(V)$ are the following ones:

(i) **spatial translations** $(x, e) \mapsto (t_v(x), e)$; they shift the centre of mass $x \in M$ along the vector $v \in V$ without affecting the internal configuration $e \in F(V)$. Analytically:

$$\left(x^i, e^i{}_A\right) \mapsto \left(x^i + v^i, e^i{}_A\right).$$

(ii) **additive translations of internal degrees of freedom:** any $\xi \in V^n$ gives rise to the mapping $(x, e) \mapsto (x, e + \xi)$, i.e.,

$$(x^i, e^i{}_A) \mapsto (x^i, e^i{}_A + \xi^i{}_A).$$

Of course, such transformations act only locally in $M \times F(V)$, because they may produce linearly dependent n-tuples of vectors from independent ones. The Hamiltonian generators are given by $P^A{}_i$.

(*iii*) **spatial affine transformations of internal degrees of freedom:** any $L \in \mathrm{GL}(V)$ acts on the configuration space as follows:

$$(x; \ldots, e_A, \ldots) \mapsto (x; \ldots, Le_A, \ldots)$$

or shortly $(x, e) \mapsto (x, Le)$. Analytically,

$$(x^i, e^i{}_A) \mapsto (x^i, L^i{}_j e^j{}_A).$$

The centre-of-mass position is not effected. The Hamiltonian generators of this group are given by $\Sigma^i{}_j$. Because of this the object $\left[\Sigma^i{}_j\right]$ is referred to as affine spin. If $g \in V^* \otimes V^*$ is the metric tensor of V, then the quantity

$$S^i{}_j = \Sigma^i{}_j - g_{jk}\Sigma^k{}_l g^{li},$$

i.e., the doubled g-skew-symmetric part of Σ, is the usual canonical spin generating rigid spatial rotations of internal (relative) degrees of freedom.

(*iv*) **material affine transformations of internal degrees of freedom:** any not singular matrix $L \in \mathrm{GL}(n, \mathbb{R})$ acts on the configuration space as follows:

$$(x; \ldots, e_A, \ldots) \mapsto (x; \ldots, e_B L^B{}_A, \ldots).$$

As in (*iii*), we use the shorthand $(x, e) \mapsto (x, eL)$. ***Important in this context:*** *do not confuse conceptually linear mappings in V with matrices!* This group is generated in the Hamiltonian sense by $\widehat{\Sigma}^A{}_B$, i.e., by co-moving components of the hypermomentum. Let us observe that the above right-acting transformations may be interpreted as ones induced by linear mappings $L \in \mathrm{GL}(n, \mathbb{R})$ acting in the "material space" $\mathbb{R}^n$. As usual, the frames themselves may be interpreted as linear mappings $e : \mathbb{R}^n \to V$. The label space $\mathbb{R}^n$ is endowed with the standard metric δ, sometimes denoted also by η, to stress the link with the usual formulation of the mechanics of extended affine systems. The δ-skew-symmetric part of $\widehat{\Sigma}^A{}_B$,

$$V^A{}_B := \widehat{\Sigma}^A{}_B - \delta_{BC}\widehat{\Sigma}^C{}_D \delta^{DA},$$

i.e., vorticity, generates rigid material rotations of the body.

(*v*) **total affine transformations in space.** They act both on the translational and internal degrees of freedom. Let $\varphi : M \to M$ be an arbitrary affine transformation acting in M, and $L[\varphi] : V \to V$ be its linear (homogeneous) part. Explicitly, for any pair of points $a, b \in M$, the radius vector $\overrightarrow{\varphi(a)\varphi(b)}$ is obtained from the vector $\overrightarrow{ab}$ through the action of $L[\varphi]$:

$$\overrightarrow{\varphi(a)\varphi(b)} = L[\varphi]\overrightarrow{ab}.$$

The action of φ on the configuration space $M \times F(V)$ is given by $(x, e) \mapsto (\varphi(x), L[\varphi]e)$. Analytically, in Cartesian coordinates x^i,

$$\left(x^i, e^i{}_A\right) \mapsto \left(L^i{}_j x^j + a^i, L^i{}_j e^j{}_A\right).$$

Infinitesimal Hamiltonian generators are given by $\left(P_i, \mathcal{J}^i{}_j\right)$, where the total canonical affine momentum $\mathcal{J}^i{}_j$ with respect to the origin of coordinates is defined as follows:

$$\mathcal{J}^i{}_j = x^i P_j + \Sigma^i{}_j,$$

i.e., it consists of the "orbital" and "internal" parts, respectively, $x^i P_j$ and $\Sigma^i{}_j$. Taking the skew-symmetric part of the above expression we obtain the usual splitting of the angular momentum onto its "orbital" and "internal" (spin) parts:

$$\Im^i{}_j = L^i{}_j + S^i{}_j,$$

where, obviously,

$$\begin{aligned} \Im^i{}_j &= \mathcal{J}^i{}_j - g^{ik} g_{jl} \mathcal{J}^l{}_k = \mathcal{J}^i{}_j - \mathcal{J}_j{}^i, \\ L^i{}_j &= x^i P_j - g^{ik} g_{jl} x^l P_k = x^i P_j - x_j P^i, \\ S^i{}_j &= \Sigma^i{}_j - g^{ik} g_{jl} \Sigma^l{}_k = \Sigma^i{}_j - \Sigma_j{}^i. \end{aligned}$$

(*vi*) **total affine transformations in the material space.** If the configuration space of extended affinely-rigid body is identified with $M \times F(V)$, then, as mentioned, the material space may be identified simply with $\mathbb{R}^n$. The current position of the constituent labelled by $a \in \mathbb{R}^n$ is displaced with respect to the instantaneous position of the centre of mass by the spatial vector u with coordinates $u^i = e^i{}_K a^K$. The affine group $\mathrm{GAf}(n, \mathbb{R})$ in $\mathbb{R}^n$ is canonically identical with the semidirect product $\mathrm{GL}(n, \mathbb{R}) \times_s \mathbb{R}^n$, i.e., with the set of pairs (B, b) multiplied (composed) according to the group rule

$$(B_1, b_1)(B_2, b_2) = (B_1 B_2, b_1 + B_1 b_2).$$

The action of (B, b) on the configuration (x, e) is given by

$$(x, e) \mapsto (t_{eb}(x), eB),$$

where in the expression eb, $e \in F(V)$ is identified with a linear isomorphism of $\mathbb{R}^n$ onto V. Analytically,

$$\left(x^i, e^i{}_A\right) \mapsto \left(x^i + e^i{}_K b^K, e^i{}_K B^K{}_A\right).$$

The corresponding system of infinitesimal Hamiltonian generators is given by $\left(\widehat{P}_A, \widehat{\Sigma}^K{}_L\right)$.

If (M, Γ) is not flat, then the above picture of the transformation groups changes in an essential way. There is no counterpart of the $n(n+1)$-dimensional group of affine transformations acting in M. In particular, there is no concept of spatial translations and radius vectors. Only the infinite-dimensional group $\mathrm{Diff}(M)$ of all diffeomorphisms of M onto M is well defined. As a rule, its elements do not preserve the parallel transport and covariant differentiation. There are no extended affinely-rigid bodies and affine degrees of freedom may be considered only as internal ones. Translations in the micromaterial space $\mathbb{R}^n$ are well defined, however, they lose the physical meaning that they had in the theory of extended affine bodies. The only transformations which survive are those dealing with internal degrees of freedom only, without any affecting of translational motion in M. Let us describe them in some details:

(i) **additive translations of internal degrees of freedom.** They act in any fibre $F_xM = \pi^{-1}(x) \subset FM$ of the bundle of frames exactly as their flat-space counterparts do in $F(V)$. There is, however, an essential novelty when these transformations are considered globally in FM, not in separate fibres. Namely, in a curved manifold there is no canonical identification between different tangent spaces. Therefore, the concept of "the same" translation in different fibres is missing, and the group becomes infinite-dimensional. Every ordered n-tuple of vector fields in M,

$$M \ni x \mapsto \xi(x) = (\ldots, \xi_A(x), \ldots) \in (T_xM)^n \,,$$

gives rise to the local transformation of FM

$$F_xM \ni e \mapsto e + \xi(x) \in F_xM \tag{1.8}$$

(local, because some frames are mapped into linearly dependent n-tuples of vectors). Obviously, the sum in the last formula is meant pointwisely, i.e., for any $A = \overline{1, n}$ the element e_A is replaced by $e_A + \xi_A(x)$. Canonical momenta $P^A{}_i$ are Hamiltonian generators of such transformations. More precisely, for any ordered n-tuple of covector fields X_A on M, the quantity $X^i{}_A P^A{}_i$ generates some one-parameter group of transformations (1.8).

(ii) **spatial affine transformations of internal degrees of freedom.** Here we are dealing with the same phenomenon as previously, namely, the n^2-dimensional Lie groups $\mathrm{GL}(T_xM)$ acting separately in tangent spaces T_xM give rise to some infinite-dimensional group of transformations acting globally in FM. This group is "parameterized" by mixed second-order tensor fields T on M, cf. (1.4), (1.5), (1.6). The quantities $\Sigma^i{}_j$ are Hamiltonian generators of this group. More precisely, for any mixed second-order tensor field α on M, the quantity $\alpha^{i}{}_{j}\Sigma^{j}{}_{i}$ generates some one-parameter group of transformations (1.4), (1.5), (1.6). There is no escape from the infinite dimension because in a curved space (M, Γ) there is no canonical identification of tangent spaces at different points.

(iii) **micromaterial affine transformations of internal degrees of freedom.** Now, in a complete analogy to the extended model in a flat space, the n^2-dimensional Lie group $\mathrm{GL}(n, \mathbb{R})$ acts on the configuration space FM according to the usual rule (1.1). The quantities $\widehat{\Sigma}^{A}{}_{B}$ are Hamiltonian generators of the corresponding Lie groups of extended point transformations.

One should stress that the quantities $L^{A}{}_{B}$ describing micromaterial transformations may be defined as constants, but they need not be so. Namely, following the pattern developed in gauge theories we can replace the matrices L by $\mathrm{GL}(n, \mathbb{R})$-valued functions on M. They act on the configuration space FM according to the rule (1.3), (1.7). Any matrix-valued function $\alpha : M \to \mathrm{L}(n, \mathbb{R})$ gives rise to the one-parameter group of transformations with the Hamiltonian generator $\alpha^{A}{}_{B}\widehat{\Sigma}^{B}{}_{A}$. In this way we obtain again, just as in previous examples, some infinite-dimensional functionally-"parameterized" transformation group.

We shall almost not deal with the diffeomorphism group $\mathrm{Diff}(M)$ or its special (volume-preserving) subgroup $\mathrm{SDiff}(M)$ as a symmetry of our model. It does not play any essential role in mechanics of material points with internal degrees of freedom. On the contrary, it is rather relevant for the theory of micromorphic continua.

Here let us only mention that any diffeomorphism $f : M \to M$ acts on the configuration space FM according to the rule

$$F_x M \ni e \mapsto Df_x \cdot e \in F_{f(x)} M,$$

i.e., analytically,

$$\left(x^i, e^{i}{}_{A}\right) \mapsto \left(f^i(x), \frac{\partial f^i}{\partial x^j}(x) e^{j}{}_{A}\right),$$

where the functions $f^i(x)$ provide analytical description of f (they express coordinates of $f(x)$ in terms of those of x).

Due to the essentially local ("x-dependent") character of transformations (1.4), (1.5), (1.7), (1.8), and admissibly local character of (1.3), their action on velocities and canonical momenta is (respectively, may be) different from that could be naively expected on the basis of analogy with the flat-space theory.

Additive translations in internal degrees of freedom (1.8) affect internal velocities as follows:

$$V^{i}{}_{A} \mapsto {}'V^{i}{}_{A} = V^{i}{}_{A} + V^{j}\nabla_{j}\xi^{i}{}_{A},$$

where, as previously, $V^{i} = dx^{i}/dt$ denotes the translational velocity in M. This rule reduces to the invariance of $(\ldots, V_{A}, \ldots)$ (characteristic for the flat-space theory) only when all vector fields ξ_{A} are parallel in the Γ-sense. Translational velocity is invariant under transformations of internal degrees of freedom, ${}'V^{i} = V^{i}$. On the contrary, the covariant canonical momenta transform according to the rules

$${}'P_{i} = P_{i} - P^{A}{}_{j}\nabla_{i}\xi^{j}{}_{A}, \qquad {}'P^{A}{}_{i} = P^{A}{}_{i}.$$

Microspatial affine transformations of internal degrees of freedom (1.4), (1.5), (1.6) also do not affect translational velocity, i.e.,

$${}'V^{i} = V^{i},$$

but the internal velocities are transformed in the following way:

$${}'V^{i}{}_{A} = T^{i}{}_{j}V^{j}{}_{A} + V^{k}\left(\nabla_{k}T^{i}{}_{j}\right)e^{j}{}_{A}. \tag{1.9}$$

This implies that

$$\begin{aligned} {}'\Omega^{i}{}_{j} &= T^{i}{}_{l}\Omega^{l}{}_{m}T^{-1m}{}_{j} + V^{k}\left(\nabla_{k}T^{i}{}_{m}\right)T^{-1m}{}_{j} \\ &= T^{i}{}_{l}\Omega^{l}{}_{m}T^{-1m}{}_{j} + \left(\nabla_{V}T^{i}{}_{m}\right)T^{-1m}{}_{j}, \end{aligned} \tag{1.10}$$

and similarly,

$${}'\widehat{\Omega}^{A}{}_{B} = \widehat{\Omega}^{A}{}_{B} + e^{A}{}_{l}T^{-1l}{}_{i}\left(\nabla_{V}T^{i}{}_{j}\right)e^{j}{}_{B}. \tag{1.11}$$

On the contrary, translational canonical momenta P_{i} suffer the transformation

$${}'P_{i} = P_{i} - \Sigma^{k}{}_{l}T^{-1l}{}_{j}\nabla_{i}T^{j}{}_{k}, \tag{1.12}$$

whereas the internal ones obey the well-known global rule:

$${}'P^{A}{}_{i} = P^{A}{}_{j}T^{-1j}{}_{i},$$

therefore,

$${}'\Sigma^{i}{}_{j} = T^{i}{}_{k}\Sigma^{k}{}_{m}T^{-1m}{}_{j}, \qquad {}'\widehat{\Sigma}^{A}{}_{B} = \widehat{\Sigma}^{A}{}_{B}.$$

The formulas (1.9), (1.10), (1.11), (1.12) reduce to the corresponding global rules from the mechanics of extended affine bodies if and only if the field T is Γ-parallel, i.e.,

$$\nabla_{k}T^{i}{}_{j} = 0.$$

The global micromaterial transformations (1.1) act just like in the flat-space theory of extended affine bodies:

$$'V^i = V^i, \qquad 'V^i{}_A = V^i{}_B L^B{}_A,$$
$$'P_i = P_i, \qquad 'P^A{}_i = L^{-1A}{}_B P^B{}_i,$$

therefore,

$$'\Omega^i{}_j = \Omega^i{}_j, \qquad '\widehat{\Omega}^A{}_B = L^{-1A}{}_C \widehat{\Omega}^C{}_D L^D{}_B,$$
$$'\Sigma^i{}_j = \Sigma^i{}_j, \qquad '\widehat{\Sigma}^A{}_B = L^{-1A}{}_C \widehat{\Sigma}^C{}_D L^D{}_B.$$

For local, i.e., x-dependent micromaterial transformations we have that

$$\begin{aligned}
'V^i &= V^i, \\
'V^i{}_A &= V^i{}_B L^B{}_A + e^i{}_B L^B{}_{A,k} V^k, \\
'\Omega^i{}_j &= \Omega^i{}_j + e^i{}_B \left(L^B{}_{A,k} L^{-1A}{}_C \right) e^C{}_j V^k, \\
'\widehat{\Omega}^A{}_B &= L^{-1A}{}_C \widehat{\Omega}^C{}_D L^D{}_B + L^{-1A}{}_C L^C{}_{B,k} V^k,
\end{aligned}$$

where comma denotes the partial differentiation. Again these formulas reduce to the preceding ones when the field L is constant, i.e., micromaterial transformations are global. By duality rule,

$$'P_i{}'V^i + '\widehat{\Sigma}^B{}_A{}'\widehat{\Omega}^A{}_B = P_i V^i + \widehat{\Sigma}^B{}_A \widehat{\Omega}^A{}_B,$$

one can easily show that

$$\begin{aligned}
'P_i &= P_i - \widehat{\Sigma}^A{}_K L^K{}_{B,i} L^{-1B}{}_A, \\
'\widehat{\Sigma}^A{}_B &= L^{-1A}{}_C \widehat{\Sigma}^C{}_D L^D{}_B, \\
'P^A{}_i &= L^{-1A}{}_B P^B{}_i.
\end{aligned}$$

Chapter 2

Metrical concepts

We have described above kinematics and canonical formalism for affine model of internal degrees of freedom in a manifold M endowed with a general affine connection Γ. The metrical concepts were yet not used at all. They become necessary when we aim at constructing dynamical models. And of course, in realistic physical problems one usually deals with some metric structure. For example, in the standard General Relativity, i.e., in the Einstein theory of gravitation, the dynamical metric tensor of the four-dimensional space-time is used for describing gravitational field. Even in more sophisticated alternative theories, like gauge models of gravitation, there exists always some physically relevant metrical aspect of the gravitational field.

So, from now on, we shall usually (nevertheless, not always) assume that the manifold M is endowed both with the metrical tensor g and affine connection Γ. In principle, one can consider the most general structure, where Γ and g are unrelated, independent on each other. Nevertheless, both from the geometrical and physical point of view of particular interest are situations when some kind of compatibility between affine and metrical structures is assumed.

Riemann-Cartan space is one in which all metrical relations like length of vectors, angles (in particular, orthogonality) are preserved by parallel transports, i.e.,

$$\nabla_k g_{ij} = 0;$$

the covariant derivative of the metric tensor vanishes. It is well known from differential geometry that then

$$\Gamma^i{}_{jk} = \left\{ \begin{array}{c} i \\ jk \end{array} \right\} + K^i{}_{jk} = \left\{ \begin{array}{c} i \\ jk \end{array} \right\} + S^i{}_{jk} + S_{jk}{}^i + S_{kj}{}^i, \tag{2.1}$$

where

$$\left\{ \begin{array}{c} i \\ jk \end{array} \right\} = \frac{1}{2} g^{im} \left(g_{mj,k} + g_{mk,j} - g_{jk,m} \right)$$

is the Levi-Civita (Christoffel) symbol,

$$S^i{}_{jk} = \Gamma^i{}_{[jk]} = \frac{1}{2} \left(\Gamma^i{}_{jk} - \Gamma^i{}_{kj} \right)$$

is the torsion tensor, and all indices are moved from their natural positions with the help of the metric tensor g. The quantity $K^i{}_{jk}$ is referred to as the contortion tensor; it satisfies

$$K^i{}_{jk} + K_j{}^i{}_k = 0.$$

The structure (M, Γ, g) reduces to the Riemann space when it is torsion-free, $S^i{}_{jk} = 0$, i.e., the object Γ is symmetric and automatically coincides with Levi-Civita symbol,

$$\Gamma^i{}_{jk} = \left\{ \begin{array}{c} i \\ jk \end{array} \right\}.$$

In this model parallelism is derived from the metric concepts.

Riemann-Cartan-Weyl space is one in which angles between vectors, but not necessarily their lengths are preserved, i.e.,

$$\nabla_k g_{ij} = -Q_k g_{ij},$$

where Q_k is referred to as the Weyl covector, and its contravariant counterpart $Q^k = g^{kj} Q_j$ as the Weyl vector. It is known from differential geometry that in Riemann-Cartan-Weyl spaces

$$\Gamma^i{}_{jk} = \left\{ \begin{array}{c} i \\ jk \end{array} \right\} + \mathcal{K}^i{}_{jk},$$

where

$$\mathcal{K}^i{}_{jk} = S^i{}_{jk} + S_{jk}{}^i + S_{kj}{}^i + \frac{1}{2} \left(\delta^i{}_j Q_k + \delta^i{}_k Q_j - g_{jk} Q^i \right)$$

with the same as previously convention concerning the raising and lowering of indices.

If Γ is symmetric, we are dealing with the Weyl space,

$$\Gamma^i{}_{jk} = \left\{ \begin{array}{c} i \\ jk \end{array} \right\} + \frac{1}{2} \left(\delta^i{}_j Q_k + \delta^i{}_k Q_j - g_{jk} Q^i \right).$$

In the micromaterial space $\mathbb{R}^n$ the standard Kronecker metric δ_{AB} is used. If for any reason we admit non-orthogonal rectilinear coordinates in $\mathbb{R}^n$, we shall use the symbol η_{AB} to denote the micromaterial metric.

For any $e \in F_xM$ the corresponding Green deformation tensor in $\mathbb{R}^n$ is given by

$$G[e]_{AB} = g_{ij}(x)e^i{}_A e^j{}_B$$

or, using the obvious abbreviations,

$$G[e] = e^* \cdot g \in \mathbb{R}^{n*} \otimes \mathbb{R}^{n*}.$$

Similarly, the Cauchy deformation tensor $C[e] \in T^*_xM \otimes T^*_xM$ is given by

$$C[e]_{ij} = \eta_{AB}e^A{}_i e^B{}_j = \delta_{AB}e^A{}_i e^B{}_j,$$

or symbolically,

$$C[e] = \widetilde{e}^* \cdot \eta.$$

The corresponding contravariant inverses will be denoted by $\widetilde{G}[e]$, $\widetilde{C}[e]$,

$$\widetilde{G}^{AC}G_{CB} = \delta^A{}_B, \qquad \widetilde{C}^{ik}C_{kj} = \delta^i{}_j.$$

Obviously,

$$\widetilde{G}[e]^{AB} = e^A{}_i e^B{}_j g(x)^{ij}, \qquad \widetilde{C}[e] = e^i{}_A e^j{}_B \eta^{AB} = e^i{}_A e^j{}_B \delta^{AB}.$$

Lagrange and Euler deformation tensors are respectively given by

$$\mathcal{E}[e] = \frac{1}{2}\left(G - \eta\right), \qquad \varepsilon[e] = \frac{1}{2}\left(g - C\right).$$

Obviously, the quantities $G[e]_{AB}$ are scalar products of vectors e_A, e_B:

$$G[e]_{AB} = g(x)\left(e_A, e_B\right) = \langle e_A | e_B\rangle = G[e]_{BA}.$$

We say that motion is metrically rigid if $\mathcal{E}$ (equivalently ε) vanishes along all admissible trajectories. In other words, only such configurations are admissible that

$$G[e]_{AB} = \eta_{AB} = \delta_{AB}.$$

Equivalently, this means that

$$C[e] = g(x), \qquad e \in F_xM.$$

Infinitesimal affinely-rigid body becomes then the infinitesimal gyroscope, i.e., the configuration space FM becomes restricted to the constraints submanifold

$F(M,g) \subset FM$ consisting of g-orthonormal frames. More precisely, one admits the connected submanifold $F^{+}(M,g)$ consisting of g-orthonormal frames positively oriented with respect to some fixed orientation in M.

As mentioned, constraints of gyroscopic motion may be described analytically by any of the following systems of equations:

$$g_{ij}e^{i}{}_{A}e^{j}{}_{B} = \eta_{AB} = \delta_{AB}, \tag{2.2}$$

$$\eta_{AB}e^{A}{}_{i}e^{B}{}_{j} = \delta_{AB}e^{A}{}_{i}e^{B}{}_{j} = g_{ij}, \tag{2.3}$$

obviously, these systems are equivalent.

Subjecting the first system to the operation D/Dt, i.e., covariant differentiation along the orbital trajectory, and contracting the resulting equation with $e^{A}{}_{i}e^{B}{}_{j}$, we obtain that

$$g_{ki}\Omega^{i}{}_{l} + g_{li}\Omega^{i}{}_{k} = -V^{m}\nabla_{m}g_{kl} = -\nabla_{V}g_{kl}. \tag{2.4}$$

Equivalently, differentiating the second system in the D/Dt-sense, we obtain the equivalent form

$$\eta_{AC}\widehat{\Omega}^{C}{}_{B} + \eta_{BC}\widehat{\Omega}^{C}{}_{A} = -\left(V^{m}\nabla_{m}g_{ij}\right)e^{i}{}_{A}e^{j}{}_{B} = -\left(\nabla_{V}g_{ij}\right)e^{i}{}_{A}e^{j}{}_{B}. \tag{2.5}$$

Let us notice that the D/Dt-differentiation of the first system is identical with the usual d/dt-differentiation, because from the point of view of geometry of M the left- and right-hand sides are scalar quantities (they are tensors in the micromaterial sense of $\mathbb{R}^{n}$).

If Γ is metrical, i.e., if (M,Γ,g) is a Riemann-Cartan space, in particular, just a Riemann space $(M,\{\},g)$, then the right-hand sides of equations (2.4), (2.5) vanish and Ω, $\widehat{\Omega}$ become respectively g- and η-skew-symmetric,

$$g_{km}\Omega^{m}{}_{l} + g_{lm}\Omega^{m}{}_{k} = 0, \qquad \eta_{KM}\widehat{\Omega}^{M}{}_{L} + \eta_{LM}\widehat{\Omega}^{M}{}_{K} = 0.$$

If the standard coordinates in $\mathbb{R}^{n}$ are used, $\eta_{AB} = \delta_{AB}$, then the last condition means simply that $\widehat{\Omega}$ is literally skew-symmetric.

Therefore, Ω, $\widehat{\Omega}$ are respectively elements of Lie algebras $\mathrm{SO}(T_{x}M,g_{x})'$, $\mathrm{SO}(n,\mathbb{R})'$ of the corresponding orthogonal groups. They are gyroscopic angular velocities respectively in the spatial and co-moving (material) representations. For a unconstrained affine motion, when Ω, $\widehat{\Omega}$ are general linear mappings, angular velocity may be defined as the corresponding g- or η-skew-symmetric part of the affine velocity, i.e.,

$$\omega^{i}{}_{j} := \frac{1}{2}\left(\Omega^{i}{}_{j} - g^{ik}g_{jl}\Omega^{l}{}_{k}\right),$$

$$\widehat{\omega}^{A}{}_{B} := \frac{1}{2}\left(\widehat{\Omega}^{A}{}_{B} - \eta^{AC}\eta_{BD}\widehat{\Omega}^{D}{}_{C}\right).$$

In the case of rigid motion they are identical respectively with $\Omega^i{}_j$ and $\widehat{\Omega}^A{}_B$. **But attention!** The general rule

$$\Omega^i{}_j = e^i{}_A \widehat{\Omega}^A{}_B e^B{}_j$$

is not valid any longer for $\omega^i{}_j$, $\widehat{\omega}^A{}_B$, i.e.,

$$\omega^i{}_j \neq e^i{}_A \widehat{\omega}^A{}_B e^B{}_j,$$

unless the motion is metrically rigid (gyroscopic), i.e., (2.2), (2.3) hold. Therefore, in non-gyroscopic motion $\widehat{\omega}^A{}_B$ fail to be co-moving components of the angular velocity. The same concerns kinematical distortions,

$$\begin{aligned} d^i{}_j &:= \frac{1}{2}\left(\Omega^i{}_j + g^{ik} g_{jl} \Omega^l{}_k\right), \\ \widehat{d}^A{}_B &:= \frac{1}{2}\left(\widehat{\Omega}^A{}_B + \eta^{AC} \eta_{BD} \widehat{\Omega}^D{}_C\right). \end{aligned}$$

As seen from the formulas (2.4), (2.5), if affine connection Γ used in the definitions of Ω and $\widehat{\Omega}$ is not metrical (i.e., (M, Γ, g) is not the Riemann-Cartan space), then Ω and $\widehat{\Omega}$ are not skew-symmetric in the g- and η-sense (i.e., are not elements of the Lie algebras of $\mathrm{SO}(T_x M, g_x)$, $\mathrm{SO}(n, \mathbb{R})$) even if motion is purely gyroscopic. Therefore, it is confusing to interpret them as angular velocities. Taking their g- and η-skew-symmetric parts also does not seem convincing. If Γ is metrical, i.e., $\nabla g = 0$, then, as mentioned, Ω and $\widehat{\Omega}$ are respectively g- and η-skew-symmetric. Nevertheless, their particular form depends explicitly on the torsion tensor S. So, no doubt, the use of gyroscopic concepts is clean only when the Levi-Civita affine connection $\{\}$ is used.

One can wonder whether gyroscopic motion could not be defined without any use of the fixed metric tensor g on M, thus, without any problems like those mentioned above (definition of angular velocity, Γ-g compatibility, and so on). Apparently, the idea might look both not physical and mathematically inconsistent. However, although the physical usefulness question is still open, the mathematical correctness may be easily shown. In a sense, gyroscopic constraints may be defined on the basis of affine connection structure (M, Γ), quite independently of the metric tensor concept. However, the micromaterial metric η_{AB} (δ_{AB}) may be used. Namely, it gives rise to the Cauchy deformation tensor

$$C_{ij} = \eta_{AB} e^A{}_i e^B{}_j$$

and its inverse

$$\widetilde{C}^{ij} = e^i{}_A e^j{}_B \eta^{AB}.$$

Obviously, $C[e] \in T^*_{\pi(e)}M \otimes T^*_{\pi(e)}M$ has all formal properties of the metric tensor in the instantaneous tangent space $T_{\pi(e)}M$ (symmetry and positive definiteness). But it is defined only at the point $x = \pi(e) \in M$ and is not induced by any metric field g living globally all over in M. And without such a field even the very term "deformation tensor" is not very adequate, because we do not have any metrical standard which might be compared with $C[e]$; thus, we cannot decide to which extent $e \in FM$ "deforms" η. Nevertheless, it is meaningful to say that some motion in FM is free of deformations when $C[e]$ is covariantly constant along the curve describing translational motion in M, i.e.,

$$\frac{DC_{ij}}{Dt} = 0.$$

After simple calculations we obtain the mutually equivalent conditions

$$\widehat{\Omega}^A{}_B = -\eta^{AK}\eta_{BL}\widehat{\Omega}^L{}_K, \tag{2.6}$$

$$\Omega^i{}_j = -\widetilde{C}^{ik}C_{jl}\Omega^l{}_k, \tag{2.7}$$

i.e., $\widehat{\Omega}$ and Ω are skew-symmetric respectively with respect to η and C used as metrics. There is however an important point, namely, if (M, Γ) is non-Euclidean (in the local sense), then the above constraints are aholonomic. If (M, Γ) is locally Euclidean, these constraints become quasi-holonomic, i.e., the manifold FM is foliated by a family of $n(n+1)/2$-dimensional integral manifolds of the above Pfaff systems. Any of these mutually disjoint manifolds is a possible configuration space of rigid motions. And any particular choice is equivalent to fixing some metric tensor field g on M. And indeed, in an n-dimensional linear space the manifold of possible metric tensors is $n(n+1)/2$-dimensional, thus, leaving $n(n-1)/2$ independent parameters in the manifold of linear frames reducing the metric to its standard Kronecker-delta form. Together with n translational degrees of freedom we obtain exactly $n(n+1)/2$ degrees of freedom of a rigid body moving in n-dimensional Euclidean space.

It is quite a different question whether the Cauchy tensor $C[e]$ may be meaningfully used as a metric-like tensor of the instantaneous tangent space $T_{\pi(e)}M$. We return later on to the question of possible physical applications. In a moment we stress interesting geometrical aspects of aholonomic constraints (2.6), (2.7).

Chapter 3

Dynamical affine models

Let us now turn to describing dynamical models. We begin from the models of kinetic energy, i.e., roughly speaking, metrics (Riemannian structures) on the configuration space FM. When we deal with internal degrees of freedom, the problem becomes delicate because the d'Alembert method of deriving the kinetic energy from the model of extended affinely constrained system becomes unjustified, not reliable, perhaps just misleading. It is so even in models of essentially internal degrees of freedom in a flat space, but in curved manifolds the problem becomes very essential, fundamentally embarrassing, and the only reliable method is one based on appropriate symmetry principle. As we saw in [49], it is the case also in some non-standard applications in the usual continuum mechanics and the dynamics of structured bodies and defects.

Nevertheless, to begin with, we discuss as first the models (metrics on FM) following formally the expressions known from the d'Alembert mechanics of extended affine bodies in flat spaces, and in any case based on simple analogies. Later on we discuss mathematical structure of more general models following the non-d'Alembert ideas in a flat space [49]. And finally, some perspectives of physical applications will be suggested and preliminarily discussed.

Repeating formally the d'Alembert expression from mechanics of affine bodies in flat spaces, we obtain that

$$T = T_{\rm tr} + T_{\rm int} = \frac{m}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} + \frac{1}{2} g_{ij} \frac{De^i{}_A}{Dt} \frac{De^j{}_B}{Dt} J^{AB}. \tag{3.1}$$

In this formula the descriptors "tr" and "int" refer obviously to the translational and internal parts, $m > 0$ is the mass, and the symmetric and positively definite micromaterial tensor $J \in \mathbb{R}^n \otimes \mathbb{R}^n$ describes the internal inertia. The main difference in comparison with the flat-space formula for extended bodies is that now the covariant along-curve derivatives of "directors" e_A are used. And there is of course one subtle point; namely, now J is not derived from the model of constrained extended system, but just postulated as something primary. In certain calculations the following equivalent expressions are convenient:

$$T_{\rm tr} = \frac{m}{2} G[e]_{AB} \widehat{V}^A \widehat{V}^B, \tag{3.2}$$

$$T_{\rm int} = \frac{1}{2} G[e]_{KL} \widehat{\Omega}^K{}_A \widehat{\Omega}^L{}_B J^{AB}. \tag{3.3}$$

Obviously, now the coefficients in quadratic forms of co-moving velocities depend on the internal configuration e. Alternatively, $T_{\rm int}$ may be expressed in spatial terms:

$$T_{\rm int} = \frac{1}{2} g_{ij} \Omega^i{}_k \Omega^j{}_l J[e]^{kl}, \tag{3.4}$$

where $J[e]$ is the configuration-dependent spatial representation of the internal inertia:

$$J[e]^{kl} = e^k{}_A e^l{}_B J^{AB}.$$

This second-order moment (quadrupole of the internal inertia) in extended body dynamics is often used in nuclear physics.
Remark: $T_{\rm int}$ in (3.1), (3.3) is invariant under translations (1.8) if and only if the field ξ of additive translations is parallel under Γ, i.e., $\nabla_{[\Gamma]}\xi = 0$.

Denoting the system of generalized coordinates of affine bodies by

$$(\ldots, q^\mu, \ldots) = (\ldots, x^i, \ldots; \ldots, e^i{}_A, \ldots),$$

and writing symbolically (3.1) in the following form:

$$T = \frac{1}{2} \mathcal{G}_{\mu\nu} \frac{dq^\mu}{dt} \frac{dq^\nu}{dt},$$

we see that the underlying Riemannian metric $\mathcal{G}$ on $F(M)$ is flat if (M, g) is locally Euclidean. It is no longer the case in a curved manifold.

In non-dissipative models with the velocity-independent Lagrangians $L = T - V(x, e)$ the resulting Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\mu} - \frac{\partial L}{\partial q^\mu} = 0 \tag{3.5}$$

may be written down in the following form:

$$m \frac{DV^a}{Dt} = \Sigma^k{}_l R^l{}_k{}^a{}_j V^j - m V^b \mathcal{K}_b{}^a{}_c V^c - \mathcal{K}_{mk}{}^a \frac{De^m{}_A}{Dt} \frac{De^k{}_B}{Dt} J^{AB} + F^a, \tag{3.6}$$

$$e^a{}_K \frac{D^2 e^b{}_L}{Dt^2} J^{KL} = -e^a{}_K g^{bm} \frac{Dg_{mc}}{Dt} \frac{De^c{}_L}{Dt} J^{KL} + N^{ab}, \tag{3.7}$$

where the meaning of symbols is as follows:

$$\begin{aligned}
\mathcal{K}^a{}_{bc} &= \Gamma^a{}_{bc} - \left\{ \begin{array}{c} a \\ bc \end{array} \right\}, \\
F^a &= g^{ab} F_b = -g^{ab} H_b V = -g^{ab} \left(\frac{\partial V}{\partial x^b} - \Gamma^i{}_{jb} e^j{}_B \frac{\partial V}{\partial e^i{}_B} \right), \\
N^{ab} &= N^a{}_c g^{cb} = -g^{bc} E^a{}_c V = -g^{bc} e^a{}_K \frac{\partial V}{\partial e^c{}_K},
\end{aligned}$$

and obviously, all shifts of tensor indices from their natural positions are meant in the sense of g.

Let us stress an important fact that in general the translational force F^a does not equal $-g^{ab}\partial V/\partial x^b$. The equality holds only when V does not depend on internal degrees of freedom, i.e., when V is a π-pull-back of some function on M, or if connection Γ is flat and local Cartesian coordinates on M are used. If V is not a π-vertical function on M, then $\partial V/\partial x^b$ is not a covariant vector in M at all, and $g^{ab}\partial V/\partial x^b$ fails to be a contravariant M-vector. Let us observe that F may be written as follows:

$$F^a = -g^{ab} \left(\frac{\partial V}{\partial x^b} + N^j{}_i \Gamma^i{}_{jb} \right).$$

The co-moving representations of F, N are given as follows:

$$\begin{aligned}
\widehat{F}_A &= F_i e^i{}_A = -H_A V, \\
\widehat{N}^A{}_B &= e^A{}_i N^i{}_j e^j{}_B = -E^A{}_B V = -e^i{}_B \frac{\partial V}{\partial e^i{}_A}.
\end{aligned}$$

Remark: the contravariant objects $\widehat{F}^A = e^A{}_i F^i$, $\widehat{N}^{AB} = e^A{}_i e^B{}_j N^{ij}$ have the forms

$$\widehat{F}^A = \widetilde{G}^{AB} \widehat{F}_B, \qquad \widehat{N}^{AB} = \widehat{N}^A{}_C \widetilde{G}^{CB},$$

i.e., the lower-case indices are raised with the help of the Green deformation tensor.

When non-potential interactions, e.g., dissipative ones, are admitted, the above general form of equations of motion in principle remains valid, however the dynamical terms F^a, N^{ab} must be replaced by more general expressions postulated on independent basis (e.g., additional friction forces linear in generalized velocities with symmetric negatively-definite coefficients matrices).

It would be technically very difficult to derive equations (3.6) directly from the second-kind Lagrange equations (3.5). It is much more convenient to use the canonical formalism and Poisson-bracket techniques. For the potential systems with Lagrangians $L = T - V\left(x^i, e^i{}_A\right)$, the Legendre transformation

$$p_\mu = \frac{\partial L}{\partial \dot{q}^\mu} = \frac{\partial T}{\partial \dot{q}^\mu} = \mathcal{G}_{\mu\nu}(q)\dot{q}^\nu$$

leads to the Hamiltonian

$$H = \mathcal{T} + V\left(x^i, e^i{}_A\right),$$

where the kinetic term

$$\mathcal{T} = \frac{1}{2}\mathcal{G}^{\mu\nu}(q)p_\mu p_\nu$$

has the form

$$\mathcal{T} = \frac{1}{2m}g^{ij}P_iP_j + \frac{1}{2}\widetilde{J}_{AB}P^A{}_iP^B{}_jg^{ij}. \tag{3.8}$$

Let us remind that $\widetilde{J}$ is the inverse of J,

$$J^{AC}\widetilde{J}_{CB} = \delta^A{}_B,$$

and the explicit expression for Legendre transformation reads that

$$P_i = \frac{\partial T}{\partial V^i} = mg_{ij}V^j, \qquad P^A{}_i = \frac{\partial T}{\partial V^i{}_A} = g_{ij}V^j{}_BJ^{BA}.$$

The formerly quoted basic Poisson brackets enable one to write down explicitly Hamiltonian equations of motion,

$$\frac{dP_i}{dt} = \{P_i, H\}, \qquad \frac{dP^A{}_i}{dt} = \{P^A{}_i, H\},$$
$$\frac{dx^i}{dt} = \{x^i, H\}, \qquad \frac{de^i{}_A}{dt} = \{e^i{}_A, H\},$$

which after some manipulations may be reduced to the form (3.6). Let us stress that the covariant derivative in (3.6) is meant in the sense of Γ, not in the g-Levi-Civita sense; an important fact to be kept in mind when they do not coincide (i.e., when (M, Γ, g) is not a Riemann space).

As expected, equations of motion (3.6) simplify in a remarkable way when (M, Γ, g) is a Riemann-Cartan space, $\nabla g = 0$. In this case $\mathcal{K}$ becomes so-called contortion,

$$\mathcal{K}^a{}_{bc} = S^a{}_{bc} + S_{bc}{}^a + S_{cb}{}^a,$$

and equations (3.6) reduce to

$$m\frac{DV^a}{Dt} = \frac{1}{2}S^k{}_l\mathcal{R}^l{}_k{}^a{}_jV^j + 2mV^bV^cS_{bc}{}^a + F^a, \qquad (3.9)$$

$$e^a{}_K\frac{D^2e^b{}_L}{Dt^2}J^{KL} = N^{ab}. \qquad (3.10)$$

In the first (translational) equation only the g-skew-symmetric part of $\Sigma^i{}_j$, i.e., $S^i{}_j$, survives, because in Riemann-Cartan spaces the curvature tensor is g-skew-symmetric in the first pair of indices. Besides of the usual external force F^a, the right-hand side of (3.9) involves two geometric forces describing the coupling between spatial geometry and kinematical quantities of the particle motion. Namely, translational velocity is quadratically coupled to the torsion, and spin is coupled to the curvature. This is in a nice way compatible with the geometric interpretation of curvature and torsion respectively in terms of rotations and translations. The geometric force

$$F^a_{\mathrm{geom}} := \frac{1}{2}S^k{}_l\mathcal{R}^l{}_k{}^a{}_jV^j + 2mV^bV^cS_{bc}{}^a$$

is so-to-speak magnetic-like in the sense that due to the corresponding anti-symmetries of R and S they are g-orthogonal to velocities and do not do any work, i.e.,

$$g_{ab}V^aF^b_{\mathrm{geom}} = 0,$$

therefore, they do not contribute to the energy balance. In particular, they do not accelerate the absolute value of V^a, $\|V\| = \sqrt{g_{ab}V^aV^b}$, but only change the direction of V^a, i.e., give rise to the bending of translational trajectory. Even in the force-free case the motion is not geodetic; there is some link between this phenomenon and geodetic deviation. Let us observe that (3.9) may be further simplified to the form

$$m\frac{D_{[g]}V^a}{Dt} = \frac{1}{2}S^k{}_l\mathcal{R}^l{}_k{}^a{}_jV^j + F^a,$$

where $D_{[g]}$ denotes the along-curve covariant differentiation in the Levi-Civita sense. However, if (M, Γ, g) is non-Riemannian, i.e., the torsion does not vanish, this simplification is a rather seeming one, because the curvature tensor $\mathcal{R}$ of Γ contains both the term corresponding to the Riemannian curvature of $\{\}$ and the terms involving torsion. And in the internal equation (3.10) there is no simplification at all. Obviously, in the purely Riemannian case, when the torsion does vanish, we obtain the maximally simple and clear system of equations of motion:

$$m\frac{DV^a}{Dt} = \frac{1}{2}S^k{}_l\mathcal{R}^l{}_k{}^a{}_jV^j + F^a, \tag{3.11}$$

$$e^a{}_K\frac{D^2e^b{}_L}{Dt^2}J^{KL} = N^{ab}, \tag{3.12}$$

and now both the covariant derivative and the curvature tensor are meant in the g-Levi-Civita sense.

It is instructive and convenient to write down these equations in terms of some balance laws. Let

$$K^a = mV^a = m\frac{dx^a}{dt},$$

$$K^{ab} = e^a{}_KV^b{}_LJ^{KL} = e^a{}_K\frac{De^b{}_L}{Dt}J^{KL}$$

denote respectively the kinematical linear momentum and kinematical affine spin (kinematical hypermomentum), just as in [49]. Then (3.11), (3.12) may be written as follows:

$$\frac{DK^a}{Dt} = \frac{1}{2m}S^k{}_l\mathcal{R}^l{}_k{}^a{}_jK^j + F^a = \frac{1}{m}K^k{}_l\mathcal{R}^l{}_k{}^a{}_jK^j + F^a, \tag{3.13}$$

$$\frac{DK^{ab}}{Dt} = \frac{De^a{}_K}{Dt}\frac{De^b{}_L}{Dt}J^{KL} + N^{ab}. \tag{3.14}$$

We by purpose use the symbols K^a, K^{ab}, not P^a, Σ^{ab}, because the latter symbols are reserved for canonical linear momentum and affine spin in contravariant representation. In the very special case of Lagrangian $L = T - V(x^i, e^i{}_A)$, where T is given as in (3.1), it is really so that these concepts are mutually identified via Legendre transformation, i.e.,

$$P^a = g^{ab}P_b = K^a = mV^a, \qquad \Sigma^{ab} = \Sigma^a{}_cg^{cb} = K^{ab}.$$

However, for more general Lagrangians and for non-d'Alembertian models of the kinetic energy T, the above relationships become false, whereas (3.13), (3.14) remain true.

The power, i.e., the time rate of work, is given by the following formula obtained by analogy with the mechanics of extended bodies [37, 49]:

$$\mathcal{P} = g_{ab}F^aV^b + g_{bc}N^{ac}\Omega^b{}_a = \mathcal{P}_{\rm tr} + \mathcal{P}_{\rm int}.$$

We can consider internal affine dynamics subject to additional constraints, just as in the case of extended affine bodies. First of all, let us consider gyroscopic constraints, i.e., assume the moving frame e to be permanently g-orthonormal. Then, as mentioned above, Ω is permanently g-skew-symmetric ($\widehat{\Omega}$ is permanently η-(δ-)skew-symmetric).

We assume here the validity of the d'Alembert model of constrained dynamics. Therefore, our equations of motion remain valid when on their right-hand sides we introduce some extra reaction forces responsible for keeping the constraints, but, roughly speaking, not influencing the along-constraints motion. Let us denote the corresponding expression by F^a_R, N^{ab}_R. They are to be added to the "true" applied dynamical quantities F^a, N^{ab}. According to the d'Alembert principle they are completely passive controls, i.e., they do not do work along any virtual motion compatible with constraints, i.e.,

$$\mathcal{P}_R = g_{ab}F^a_R V^b + g_{bc}N^{ac}_R \Omega^b{}_a = 0$$

for any V and for any g-skew-symmetric Ω. This means that $F^a_R = 0$ and $N^{ab}_R = N^{ba}_R$, i.e., F_R vanishes and N_R is symmetric. Therefore, the effective, reaction-free system of equations of motion consists of (3.13) and the skew-symmetric part of (3.14) with algebraically substituted gyroscopic constraints $g_{ij}e^i{}_A e^j{}_B = \eta_{AB} = \delta_{AB}$, i.e.,

$$\begin{aligned}\frac{DK^a}{Dt} &= \frac{1}{2m}S^k{}_l \mathcal{R}^l{}_k{}^a{}_j K^j + F^a,\\ \frac{DS^{ab}}{Dt} &= \mathcal{N}^{ab},\end{aligned} \tag{3.15}$$

where, obviously,

$$S^{ab} = K^{ab} - K^{ba}$$

is the internal angular momentum (spin) and

$$\mathcal{N}^{ab} = N^{ab} - N^{ba}$$

is the skew-symmetric moment, torque, i.e., generalized force coupled to rotational degrees of freedom. In this way one obtains the system of $n(n+1)/2$ equations of motion imposed on the time-dependence of $n(n+1)/2$ degrees of freedom of the infinitesimal gyroscope moving in M (n translational degrees of freedom and $n(n-1)/2$ gyroscopic ones).

Returning to the explicit description in terms of the configuration variables we obtain that

$$\begin{aligned}m\frac{DV^a}{Dt} = m\frac{D^2x^a}{Dt^2} &= \frac{1}{2}S^k{}_l \mathcal{R}^l{}_k{}^a{}_j \frac{dx^j}{dt} + F^a,\\ e^a{}_K \frac{D^2 e^b{}_L}{Dt^2} J^{KL} - e^b{}_K \frac{D^2 e^a{}_L}{Dt^2} J^{KL} &= \mathcal{N}^{ab} = N^{ab} - N^{ba}.\end{aligned} \tag{3.16}$$

In these equations only given (reaction-free) forces and moments are present and obviously

$$S^{ij} = K^{ij} - K^{ji} = e^{i}{}_{A}\frac{De^{j}{}_{B}}{Dt}J^{AB} - e^{j}{}_{A}\frac{De^{i}{}_{B}}{Dt}J^{AB}.$$

In analogy to extended affine bodies one can also consider internal dynamics with other kinds of constraints. The most natural ones are those with the lucid group-theoretical structure. For example, for incompressible bodies, (3.12), (3.14) are to be replaced by their g-traceless parts:

$$e^{a}{}_{K}\frac{D^2e^{b}{}_{L}}{Dt^2}J^{KL} - \frac{1}{n}g_{cd}e^{c}{}_{K}\frac{D^2e^{d}{}_{L}}{Dt^2}g^{ab}J^{KL} = N^{ab} - \frac{1}{n}g_{cd}N^{cd}g^{ab},$$

i.e.,

$$\frac{D}{Dt}\left(K^{ab} - \frac{1}{n}g_{cd}K^{cd}g^{ab}\right) = N^{ab} - \frac{1}{n}g_{cd}N^{cd}g^{ab}$$
$$+\frac{De^{a}{}_{K}}{Dt}\frac{De^{b}{}_{L}}{Dt}J^{KL} - \frac{1}{n}g_{cd}\frac{De^{c}{}_{K}}{Dt}\frac{De^{d}{}_{L}}{Dt}J^{KL}g^{ab}.$$

Incompressibility constraints may be described analytically by equations

$$\det\left[e^{A}{}_{i}\right] = \sqrt{\det\left[g_{ij}\right]},$$

or infinitesimally in any of two equivalent forms

$$\Omega^{i}{}_{i} = 0, \qquad \widehat{\Omega}^{A}{}_{A} = 0.$$

Let us recall that gyroscopic constraints in Riemannian space may be also described in terms of equivalent infinitesimal conditions:

$$\Omega^{i}{}_{j} = -\Omega_{j}{}^{i} = -g^{ik}g_{jl}\Omega^{l}{}_{k}, \qquad \widehat{\Omega}^{A}{}_{B} = -\widehat{\Omega}_{B}{}^{A} = -\eta^{AC}\eta_{BD}\widehat{\Omega}^{D}{}_{C}.$$

If only purely dilatational internal motion is admitted, i.e.,

$$\Omega^{i}{}_{j} = \lambda\delta^{i}{}_{j}, \qquad \widehat{\Omega}^{A}{}_{B} = \lambda\delta^{A}{}_{B},$$

or, equivalently,

$$\Omega^{i}{}_{j} - \frac{1}{n}\Omega^{k}{}_{k}\delta^{i}{}_{j} = 0, \qquad \widehat{\Omega}^{A}{}_{B} - \frac{1}{n}\widehat{\Omega}^{C}{}_{C}\delta^{A}{}_{B} = 0,$$

then the internal motion is (on the basis of d'Alembert principle) completely described by the g-trace of (3.12) or (3.14):

$$g_{ab}e^{a}{}_{K}\frac{D^2 e^{b}{}_{L}}{Dt^2}J^{KL} = g_{ab}N^{ab}, \tag{3.17}$$

$$g_{ab}\frac{DK^{ab}}{Dt} = g_{ab}\frac{De^{a}{}_{K}}{Dt}\frac{De^{b}{}_{L}}{Dt}J^{KL} + g_{ab}N^{ab}. \tag{3.18}$$

If the body is subject to Weyl constraints, i.e., its internal degrees of freedom undergo only rigid rotations and dilatations, then equations of internal motion are given by (3.15) and (3.18), or equivalently (3.16) and (3.17).

A very interesting example of constraints is that of rotation-less motion in M,

$$\Omega^{i}{}_{j} = \Omega_{j}{}^{i} = g_{jk}g^{il}\Omega^{k}{}_{l}. \tag{3.19}$$

Then the d'Alembert principle implies that equations of internal motion (effective ones, free of reaction forces) are given by the symmetric part of (3.12) or (3.14), i.e.,

$$e^{a}{}_{K}\frac{D^2 e^{b}{}_{L}}{Dt^2}J^{KL} + e^{b}{}_{K}\frac{D^2 e^{a}{}_{L}}{Dt^2}J^{KL} = N^{ab} + N^{ba},$$

i.e.,

$$\frac{D}{Dt}\left(K^{ab} + K^{ba}\right) = \frac{De^{a}{}_{K}}{Dt}\frac{De^{b}{}_{L}}{Dt}J^{KL} + \frac{De^{b}{}_{K}}{Dt}\frac{De^{a}{}_{L}}{Dt}J^{KL} + N^{ab} + N^{ba}.$$

We have mentioned that when the geometry of M is non-Euclidean (curved), then in a natural way aholonomic velocities and other aholonomic concepts appear. In the last example situation is even much more complicated, because, as mentioned in our papers about extended affine bodies [36, 37], constraints (3.19) are very essentially aholonomic even in mechanics of extended affine bodies in flat spaces; the more so in a non-Euclidean manifold.

From now on we concentrate on the mechanics of infinitesimal gyroscopes and infinitesimal affine bodies without additional constraints, but very often with the special and strong stress on the mutual coupling between rotational and deformative motion. This induces us to develop certain analytical procedures. They are even more than analytical procedures, because the underlying geometric techniques are interesting in themselves and simultaneously they reveal certain very interesting mechanical facts.

So, let us begin with the special case of infinitesimal gyroscope in a Riemann space $(M, \{\}, g)$, where equations of motion are given by (3.15) or, more explicitly, by (3.16).

Let us stress an important fact. In mechanics of unconstrained infinitesimal affine bodies, equations of motion (3.11), (3.12) or (3.13), (3.14) are directly applicable even in the purely technical sense, because $\left(x^i, e^i{}_A\right)$ are "good" independent (unconstrained) generalized coordinates. It is also so in expressions (3.1), (3.8) for the kinetic energy. However, after the gyroscopic constraints are imposed,

$$g_{ij} e^i{}_A e^j{}_B = \eta_{AB} \, (= \delta_{AB}) \, ,$$

the quantities $e^i{}_A$ are no longer independent and cannot be used as generalized coordinates. Even when we study the mutual coupling of rotational and deformative degrees of freedom, the quantities $e^i{}_A$ are inconvenient, although well defined as generalized coordinates. In flat spaces the procedure is obvious: the system $[e^i{}_A]$ as a matrix is subject to various polar, two-polar [49], and similar decompositions, and then the mutual interplay between rotations and deformations is easily treatable. Something similar must be done here, but the direct methods based on the flat-space geometry are not applicable any longer. Instead, some geometric techniques based on orthonormal aholonomic reference frames may be developed. In the special case of gyroscopic constraints one can use then various well-known coordinates on the special orthogonal group $\mathrm{SO}(n, \mathbb{R})$ as a subsystem of well-defined generalized coordinates on (FM, g).

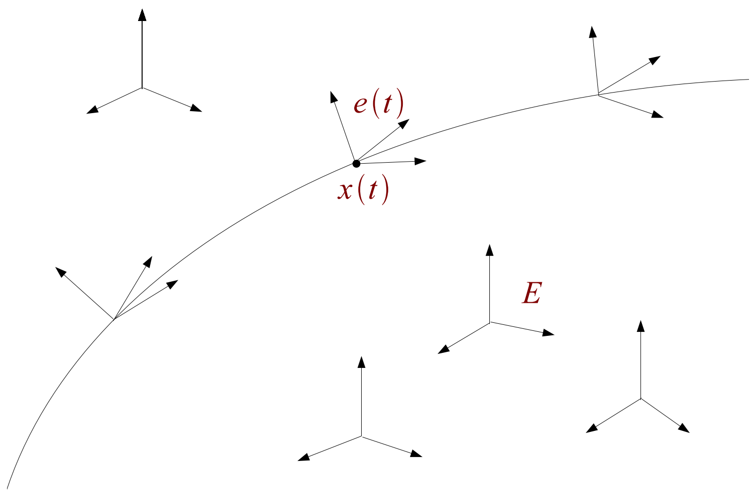

Fig. 3

So, let E be some preestablished, fixed once for all fields of linear orthonormal frames on M, i.e., a cross-section of the subbundle $(FM, g) \subset FM$ over

M. By the way, the orthonormality demand is a technically simplifying one, but in certain problems some general frames would do. But let us assume E to be orthonormal in the g-sense, i.e.,

$$g(E_A, E_B) = g_{ij} E^i{}_A E^j{}_B = \eta_{AB} = \delta_{AB}.$$

The dual co-frame $\widetilde{E} = (\ldots, E^A, \ldots)$ will be also used,

$$\langle E^A, E_B \rangle = E^A{}_i E^i{}_B = \delta^A{}_B.$$

It is obviously orthonormal with respect to the reciprocal (contravariant) metric $\widetilde{g}$ on M,

$$E^A{}_i E^B{}_j g^{ij} = \eta^{AB} \left(= \delta^{AB}\right).$$

Remark: this is a field of frames, and it is defined all over in M and kept fixed. The natural question arises as to the choice of this aholonomic frame in M. In general the particular structure of the Riemann space (M, g) suggests some choices, both technically convenient and geometrically lucid, which are well suited to the problem.

So, M is "inhabited" by continuum of orthonormal frames. The metric field on M may be expressed as follows:

$$g = \eta^{AB} E_A \otimes E^B = \delta^{AB} E_A \otimes E^B.$$

In the course of time the body moves in M. At the time instant $t \in \mathbb{R}$ it is instantaneously placed at the geometric point $x(t) \in M$ and its internal configuration is then given by $e(t) \in F_{x(t)}M$. Obviously, the vectors $e_A(t) \in T_{x(t)}M$, $A = \overline{1, n}$, may be expanded with respect to the just passed frame $E_{x(t)} \in F_{x(t)}M$, i.e.,

$$e_A(t) = E_{x(t)B} R^B{}_A(t),$$

where, in the gyroscopic case, the matrix R is orthogonal,

$$\delta_{AB} = \delta_{CD} R^C{}_A R^D{}_B, \tag{3.20}$$

or if for any reason we use a more general (non-standard) basis in $\mathbb{R}^n$, then

$$\eta_{AB} = \eta_{CD} R^C{}_A R^D{}_B.$$

Roughly, (3.20) may be written in the usual matrix terms:

$$R^T R = I_n,$$

where the $n \times n$ identity matrix is on the right-hand side. In this way the configuration space $F(M,g)$, or more precisely $F^{+}(M,g)$ (orthonormal frames positively oriented with respect to some fixed standard of orientation) is identified by the field E with the product manifold

$$M \times \mathrm{SO}(n,\mathbb{R}).$$

Obviously, this diffeomorphism depends explicitly on the choice of E. And of course everything is based on some topological assumption about M, namely, that it admits a globally defined smooth field of frames (and then, obviously, infinity of them). Instead of describing motion in terms of time-dependent quantities $x^{i}(t)$, $e^{i}{}_{A}(t)$, we describe it in terms of quantities $x^{i}(t)$, $R^{A}{}_{B}(t)$. In other words, the motion is represented by a pair of curves: the one in M, i.e., the translational motion, and the other in $\mathrm{SO}(n,\mathbb{R})$-internal rotational motion. And let us repeat that this trivialization (splitting) is possible only when the reference field E is fixed and depends explicitly on it. One of the main advantages is a possibility of introducing good generalized coordinates instead of redundant quantities $(x^{i}, e^{i}{}_{A})$. Namely, $\mathrm{SO}(n,\mathbb{R})$ admits many well-investigated and geometrically convenient coordinatizations like, first of all, canonical coordinates of the first kind (components of the rotation vector if $n=3$), canonical coordinates of the second kind, and also some other possibilities. For example, in the physically interesting special case one uses also Euler angles or spherical variables in the space of rotation vectors. So, in any case we have at disposal various natural choices of "good", i.e., not redundant generalized coordinates $(\ldots, x^{i}, \ldots; \ldots, \xi^{\mu}, \ldots)$, where ξ^{μ}, $\mu = \overline{1, n(n-1)/2}$, are the mentioned coordinates on $\mathrm{SO}(n,\mathbb{R})$.

The same procedure may be applied in the general case of the infinitesimal affinely-rigid body. The prescribed field of reference frames $E = (\ldots, E_{A}, \ldots)$ may be in principle quite general, but usually it is chosen as g-orthonormal, even if deformations are admitted. The reason is that such a description is physically convenient when dealing with mutual interactions of rigid rotations and finite or infinitesimal homogeneous deformations. Just as previously we expand

$$e_{A}(t) = E_{x(t)B}\varphi^{B}{}_{A}(t), \qquad \varphi^{A}{}_{B}(t) = \left\langle E^{A}, e_{B}(t)\right\rangle,$$

but now $\varphi(t) \in \mathrm{GL}(n,\mathbb{R})$ is a general non-singular matrix, or usually a general positive-determinant matrix, $\varphi(t) \in \mathrm{GL}^{+}(n,\mathbb{R})$. In analytical procedures describing realistic and simple isotropic problems we often represent φ in terms of the polar or two-polar decomposition [49].

In this way the bundle of linear frames FM $(F^{+}M)$ is represented by a trivialization $M \times \mathrm{GL}(n,\mathbb{R})$ $(M \times \mathrm{GL}^{+}(n,\mathbb{R}))$, also explicitly dependent on the choice of the frame E. Analytically, configurations are parameterized by generalized coordinates $(\ldots, x^i, \ldots; \ldots, \varphi^A{}_B, \ldots)$. Let us stress once again that now it is not logically necessary, because the quantities $(\ldots, x^i, \ldots; \ldots, e^i{}_A, \ldots)$ themselves are good, independent generalized coordinates. The point is only that that, as mentioned, in many realistic problems generalized coordinates $(\ldots, x^i, \ldots; \ldots, \varphi^A{}_B, \ldots)$ are geometrically and technically more convenient and computationally effective. The more so it is when $\varphi^A{}_B$ are replaced by coordinates appearing in the polar and two-polar decompositions of φ.

Let us begin with the general description of affinely-rigid body in terms of prescribed reference frame E and later on consider the special case of infinitesimal gyroscopes. This will be also convenient for making the explicit use of the polar and two-polar decompositions, in particular, for discussing integrable and superintegrable models in a two-dimensional space. The latter problem, i.e., two-dimensional infinitesimal gyroscopes or homogeneously deformable ones sliding over two-dimensional curved manifolds may be useful in the theory of microstructured (micropolar or micromorphic) plates.

Obviously, the primary dynamical variables are the moving-frame vectors $e_A(t)$, or equivalently the moving-frame covectors, $e^A(t)$, whereas the reference frames E are only the auxiliary quantities, does not matter how important for computational purposes. Let us express

$$e_A(t) = E_B(x(t))\varphi^B{}_A(t), \qquad e^A(t) = \varphi^{-1A}{}_B(t)E^B(x(t)), \tag{3.21}$$

where, obviously, $e(t) \in F_{x(t)}M$, i.e., $x(t) = \pi(e(t))$.

We shall also need inverse formulas, i.e.,

$$E_A(x(t)) = e_B(t)\varphi^{-1B}{}_A(t), \qquad E^A(x(t)) = \varphi^A{}_B(t)e^B(t). \tag{3.22}$$

To avoid unnecessary crowd of symbols, when it is not confusing, we shall omit the arguments $t, x(t)$ in the above quantities.

To express explicitly affine velocities we must find formulas for the covariant derivatives De_A/Dt in terms of the quantities E, φ.

The above expressions (3.21) imply that the covariant differentiation of $\mathbb{R} \ni t \mapsto e_A \in FM$ along the curve of translational motion $\mathbb{R} \ni t \mapsto x(t) \in M$ is given by

$$\frac{De_A}{Dt} = \frac{D}{Dt}\left(E_B\varphi^B{}_A\right) = \frac{DE_B}{Dt}\varphi^B{}_A + E_B\frac{d\varphi^B{}_A}{dt}. \tag{3.23}$$

Let us express the along-curve differentiation of E_B through its field-differentiation (well defined because E is defined as a field all over in M),

$$\frac{DE_B}{Dt} = (\nabla_i E_B)\frac{dx^i}{dt} = (\nabla_C E_B)\, E^C{}_i\frac{dx^i}{dt}, \tag{3.24}$$

where $\nabla_C E_B$ is an abbreviation for $\nabla_{E_C} E_B$, and let us remind that for any vector field Y and tensor field T, $\nabla_Y T$ denotes the covariant derivative of T along Y. Analytically,

$$\nabla_Y T = Y^i \nabla_i T.$$

It is convenient to use aholonomic coefficients of our affine connection with respect to the field E,

$$\nabla_C E_B = \Gamma^A{}_{BC} E_A. \tag{3.25}$$

The usual holonomic coefficients of Γ with respect to coordinates x^i are given by the expression

$$\Gamma^i{}_{jk} = E^i{}_A \Gamma^A{}_{BC} E^B{}_j E^C{}_k + E^i{}_A E^A{}_{j,k},$$

where the comma as usual denotes the partial differentiation. The second term

$$\Gamma[E]^i{}_{jk} := E^i{}_A E^A{}_{j,k} \tag{3.26}$$

denotes the teleparallelism connection induced by E. It is defined as follows:

$$\nabla_{[\Gamma[E]]} E_A = 0, \tag{3.27}$$

i.e., it is the only affine connection with respect to which all the fields E_A (of course, also their dual co-fields E^A) are parallel. Its curvature tensor vanishes and the torsion equals

$$S[E]^i{}_{jk} = E^i{}_A E^A{}_{[j,k]} = \frac{1}{2} E^i{}_A \left(E^A{}_{j,k} - E^A{}_{k,j} \right). \tag{3.28}$$

The parallel transport with respect to $\Gamma[E]$ is path-independent. A tensor field T is $\Gamma[E]$-parallel, $\nabla_{[\Gamma[E]]} T = 0$, if its aholonomic components with respect to E are constant in M,

$$\begin{aligned} T^{i_1 \dots i_k}{}_{j_1 \dots j_l} &= T^{I_1 \dots I_k}{}_{J_1 \dots J_l} E_{I_1} \otimes \cdots \otimes E_{I_k} \otimes E^{J_1} \otimes \cdots \otimes E^{J_l}, \\ T^{I_1 \dots I_k}{}_{J_1 \dots J_l} &= \text{const.} \end{aligned} \tag{3.29}$$

If the field of frames E is holonomic, in particular, if it is given by the system of tangent vectors $\partial/\partial x^i$ of x^i, then

$$\nabla_{\partial/\partial x^k} \frac{\partial}{\partial x^j} = \Gamma^i{}_{jk} \frac{\partial}{\partial x^i},$$

which means that we obtain the usual components of Γ with respect to local coordinates x^i.

Obviously, the quantity

$$\Gamma^i{}_{jk} - \Gamma[E]^i{}_{jk} = E^i{}_A \Gamma^A{}_{BC} E^B{}_j E^C{}_k,$$

i.e., in the coordinate-free form

$$\Gamma - \Gamma[E] = \Gamma^A{}_{BC} E_A \otimes \widetilde{E}^B \otimes \widetilde{E}^C$$

is a tensor field, once contravariant and twice covariant, as it is always the case with the difference of affine connections.

The teleparallelism torsion is directly related to what is well known in differential geometry as the aholonomic object of E, i.e.,

$$S[E] = \frac{1}{2} \Omega^A{}_{BC} E_A \otimes \widetilde{E}^B \otimes \widetilde{E}^C.$$

In other words the doubled E-co-moving components of $S[E]$ coincide with the Schouten aholonomic complex Ω; the latter may be defined in terms of Lie brackets of basic vector fields,

$$\Omega^A{}_{BC} = 2\widehat{S}^A{}_{BC} = \left\langle E^A, [E_B, E_C] \right\rangle, \tag{3.30}$$

i.e.,

$$[E_A, E_B] = \widehat{\Omega}^C{}_{AB} E_C, \qquad dE^A = \frac{1}{2} \widehat{\Omega}^A{}_{BC} E^C \wedge E^B. \tag{3.31}$$

After some calculations performed on (3.23) with the use of (3.21), (3.22), (3.24), (3.25), we finally obtain the formula

$$\frac{De_A}{Dt} = e_B \widehat{\Omega}^B{}_A,$$

where the co-moving affine velocity $\widehat{\Omega}$ may be expressed as follows:

$$\widehat{\Omega}^B{}_A = \varphi^{-1B}{}_F \Gamma^F{}_{DC} \varphi^D{}_A \varphi^C{}_E \widehat{V}^E + \varphi^{-1B}{}_C \frac{d\varphi^C{}_A}{dt}, \tag{3.32}$$

where, as previously,

$$\widehat{V}^K = e^K{}_i \frac{dx^i}{dt} = e^K{}_i V^i$$

denotes the co-moving components of translational velocity. In this way the quantity

$$\widehat{\Omega}^B{}_A = \left\langle e^B, \frac{De_A}{Dt} \right\rangle = e^B{}_i \frac{De^i{}_A}{Dt}$$

is expressed through the matrix φ and its time derivatives. This representation becomes very convenient when gyroscopic constraints are imposed. Let us observe that in addition to the second term $\varphi^{-1}d\varphi/dt$ familiar from the flat-space theory, the above expression (3.32) contains the first term explicitly depending on the geometry of M (more precisely, on (M, Γ)) and on the choice of the auxiliary reference frame E. Roughly speaking, the quantity $\varphi^{-1}d\varphi/dt$ is the co-moving affine velocity of internal degrees of freedom, as seen from the point of view of the just passed frame $E_{x(t)}$. And the first term represents the contribution to $\widehat{\Omega}$ coming from the rotation-deformation of E itself.

It is interesting to rewrite the equations (3.9), (3.10), (3.11), (3.12), and (3.13), (3.14) in purely co-moving terms. After some calculations we obtain that

$$
\begin{aligned}
m\frac{d\widehat{V}^A}{dt} &= -m\widehat{\Omega}^A{}_B\widehat{V}^B + 2m\widehat{V}^B\widehat{V}^C S_{BC}{}^A \\
&\quad + \frac{1}{2}\widehat{\Omega}^C{}_D R^D{}_C{}^A{}_B\widehat{V}^B + \widehat{F}^A, \qquad (3.33)\\
\frac{d\widehat{\Omega}^D{}_C}{dt}J^{CA} &= -\widehat{\Omega}^B{}_D\widehat{\Omega}^D{}_C J^{CA} + \widehat{N}^{AB}. \qquad (3.34)
\end{aligned}
$$

Obviously, the torsion here is taken into account, so strictly speaking they are co-moving counterparts of (3.9), (3.10).

Remark: do not confuse the three-index torsion tensor with the two-index spin one. The two-index quantity in the curvature term is vorticity.

Using explicitly the balance form of the equations of motion for the linear momentum and affine spin we obtain that

$$
\begin{aligned}
\frac{d\widehat{P}^A}{dt} &= -\widehat{P}^B\widetilde{J}_{BC}\widehat{K}^{CA} + \frac{2}{m}\widehat{P}^B\widehat{P}^C S_{BC}{}^A \\
&\quad + \frac{1}{2m}\widehat{K}^C{}_D R^D{}_C{}^A{}_B\widehat{P}^B + \widehat{F}^A, \qquad (3.35)\\
\frac{d\widehat{K}^{AB}}{dt} &= -\widehat{K}^{AC}\widetilde{J}_{CD}\widehat{K}^{DB} + \widehat{N}^{AB}, \qquad (3.36)
\end{aligned}
$$

where, as previously, $\widetilde{J}_{BC}J^{CA} = \delta^A{}_B$.

Obviously, we are allowed to use the usual time derivatives at the left-hand side of equations because $\widehat{P}^A$, $\widehat{K}^{AB}$, $\widehat{V}^A$, $\widehat{\Omega}^A{}_B$ are scalars from the point of view of geometry of M (although they are tensors in the micromaterial space $\mathbb{R}^n$).

These are, so to speak, affine Euler equations. By the way, when we impose metrical constraints, i.e., assume that permanently

$$g(e_A, e_B) = g_{ij}e^i{}_A e^j{}_B = \delta_{AB}, \qquad \varphi \in \mathrm{SO}(n, \mathbb{R}), \qquad (3.37)$$

and according to the d'Alembert principle the reaction-free equations of internal motion, i.e., the skew-symmetric part of (3.36) is only left, then we exactly obtain Euler equations for infinitesimal gyroscope in a non-Euclidean space.

Let us observe that all the above equations of motion, in particular (3.9), (3.10) and the expression for the kinetic energy (3.1), or its equivalent form (3.2), (3.3) may be expressed in terms of (3.32), and in the case of gyroscopic motion this becomes technically unavoidable. The point is that $e^{i}{}_{A}$ are not then independent generalized quantities, just due to the constraints (3.37). And although equations of motion in their general balance form (3.15) are correct and free of reaction forces, in particular dynamical problems they are not directly useful in analytical calculations and analysis of the phase portraits. The quantities $\varphi^{K}{}_{L}$ are not generalized coordinates either, because the matrix φ is orthogonal. However, as mentioned, $\varphi \in \mathrm{SO}(n,\mathbb{R})$ may be easily parameterized on the basis of group-theoretical considerations, and the corresponding parameters are just proper generalized coordinates to be effectively used in equations of motion. Let q^{α}, $\alpha = \overline{1, n(n-1)/2}$, be such group parameters. Then taking some independent subsystem of (3.15), e.g., one given by $a < b$ (or conversely), we obtain a system of $n(n-1)/2$ second-order differential equations imposed on the time-dependence of $n(n-1)/2$ generalized coordinates. In variational problems it is more convenient to substitute the expressions $\varphi(q)$ directly to the formula for the kinetic energy (3.1) or its Hamiltonian version (3.6), and then obtain directly the corresponding Euler-Lagrange equations or (more convenient) Hamilton ones formulated in terms of Poisson brackets. Such an approach is better suited to the study for integrability problems and action-angle variables. Geometrically the most suggestive choice of generalized coordinates q^{α} is that based on the exponential representation of φ. The quantities q^{α} are then canonical coordinates of the first kind on the group manifold of $\mathrm{SO}(n,\mathbb{R})$. More precisely, let $M_{KL} \in \mathrm{SO}(n,\mathbb{R})'$ be basic skew-symmetric matrices; they have only two not vanishing entries, namely ± 1 in the rows and columns labelled by K, L [49]. Then $R \in \mathrm{SO}(n,\mathbb{R})$ may be represented as follows:

$$R(\varepsilon) = \exp\left(\frac{1}{2}\varepsilon^{KL} M_{LK}\right),$$

where ε is an arbitrary real skew-symmetric matrix, $\varepsilon^{KL} = -\varepsilon^{LK}$, and, e.g., ε^{KL} with $K < L$ may be chosen as independent coordinates. The corresponding M_{KL}, $K < L$, form a basis of the Lie algebra $\mathrm{SO}(n,\mathbb{R})'$.

In the special case of $n = 3$, due to the exceptional isomorphism between axial vectors and skew-symmetric second-order tensors in $\mathbb{R}^3$, it is customary to write [49]

$$R\left(\overline{k}\right) = \exp\left(k^{A} M_{A}\right),$$

where M_A, $A = 1, 2, 3$, are basic skew-symmetric matrices,

$$(M_A)_{BC} = -\varepsilon_{ABC}$$

(ε is the totally skew-symmetric Levi-Civita-Ricci symbol normalized by $\varepsilon_{123} = 1$), and k^A are components of the so-called rotation vector in $\mathbb{R}^3$. Its length

$$k = \sqrt{(k^1)^2 + (k^2)^2 + (k^3)^2}$$

satisfies $k \leq \pi$ if all possible direction versors $\overline{n} = \overline{k}/k$ are admitted. Obviously, for any versor $\overline{n}$ we have

$$R\left(\pi\overline{n}\right) = R\left(-\pi\overline{n}\right).$$

Therefore, the group manifold is represented in the space of $\mathbb{R}^3$-vectors $\overline{k}$ by the ball of radius π, with the proviso that antipodal points on the limiting sphere $k = \pi$ are identified. This provides a very nice form of understanding that the group $\mathrm{SO}(3, \mathbb{R})$ is doubly connected.

In the three-dimensional case one uses also other parameterizations, depending on dynamical details of the considered problem. For example, in certain dynamical models of the spherical rigid body it is convenient to use the spherical variables (k, ϕ, θ) in the space of the rotation vectors $\overline{k}$ [49]:

$$k^1 = k \sin\theta \cos\phi, \qquad k^2 = k \sin\theta \sin\phi, \qquad k^3 = k\cos\theta.$$

In the theory of a free or heavy symmetric top ($I_1 = I_2$, but in general $I_1 \neq I_3$), it is convenient to use Euler angles $(\varphi, \vartheta, \psi)$,

$$R[\varphi, \vartheta, \psi] = \exp(\varphi M_3) \exp(\vartheta M_1) \exp(\psi M_3).$$

One can also use canonical coordinates of the second kind (α, β, γ), i.e.,

$$R[\alpha, \beta, \gamma] = \exp(\alpha M_1) \exp(\beta M_2) \exp(\gamma M_3).$$

Surprisingly enough, for the spherical rigid body ($I_1 = I_2 = I_3$) the kinetic energy expressions have almost the same form in coordinates $(\varphi, \vartheta, \psi)$ and (α, β, γ).

If $n = 2$, the situation is much more simple, because then $\mathrm{SO}(2, \mathbb{R})$ is one-dimensional (thus, obviously, commutative) and $R(\varphi) = \exp(\varphi\epsilon)$, where $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = -\epsilon_{21} = 1$, i.e., explicitly

$$R(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}.$$

In the case of rigid motion the second term in (3.32), $\varphi^{-1}d\varphi/dt$ is always skew-symmetric, i.e., an element of the Lie algebra $\mathrm{SO}(n,\mathbb{R})'$. More precisely, if we use artificial not orthogonal coordinates in $\mathbb{R}^3$, it is η-skew-symmetric, i.e.,

$$\varphi^{-1B}{}_C \frac{d\varphi^C{}_A}{dt} = -\eta_{AK}\eta^{BL}\varphi^{-1K}{}_C \frac{d\varphi^C{}_L}{dt}.$$

If the connection Γ is metrical, $\nabla_{[\Gamma]}g = 0$ (Riemann-Cartan space), then, as we saw, the total $\widehat{\Omega}$ is also skew-symmetric (η-skew-symmetric) and therefore, so is the first term of (3.32). By the way, the aholonomic connection coefficients $\Gamma^F{}_{DC}$ are then also skew-symmetric (more precisely, η-skew-symmetric). Just as in the general affine motion, we can say that $\widehat{\Omega}$ is then the co-moving angular velocity, $\varphi^{-1}d\varphi/dt$ is the angular velocity with respect to the pre-fixed aholonomic frame E, and the first term is, in a sense, the angular velocity with which E itself rotates along the trajectory of the structured material point.

In both the general affine and constrained gyroscopic motion the expression of kinetic energy through the above quantities becomes relatively lucid and computationally effective when the manifold (M,Γ,g) has some special structure, e.g., if it is a constant-curvature space and if the auxiliary field of frames E is appropriately chosen. The proper choice depends on the particular geometry of M, and it is also a matter of some inventive intuition. In such special cases the problem may be effectively studied on the rigorous analytical level and, in particular, interesting results concerning integrability and degeneracy (superintegrability) may be obtained [13, 14, 15, 16, 17, 18, 24, 25, 26, 27].

Unless otherwise stated, from now on we concentrate on Riemannian spaces, when Γ is the Levi-Civita connection built of g. The general case, when Γ, g are unrelated may be also interesting in itself, but certainly there are some difficulties to be overcome, because even in the rigid motion the affine velocity Ω is not g-skew-symmetric and its co-moving representation $\widehat{\Omega}$ is not η-(δ-)skew-symmetric. The fact that they do not belong to the corresponding Lie algebras of the rotational groups $\mathrm{SO}(T_xM, g_x)$, $\mathrm{SO}(n,\mathbb{R})$ obscures their interpretation. If (M,Γ,g) is a Riemann-Cartan space but Γ is not symmetric (i.e., it is not Levi-Civita connection), then affine velocities are skew-symmetric (respectively in a g- or η-sense), but they are explicitly dependent on the torsion tensor. This also leads to certain interpretation problems. Namely, one obtains different expressions for the kinetic energy and different equations of motion when one uses respectively the affine connections $\left\{ \begin{array}{c} i \\ jk \end{array} \right\}$ (g-Levi-Civita) and

$$\Gamma^i{}_{jk} = \left\{ \begin{array}{c} i \\ jk \end{array} \right\} + S^i{}_{jk} + S_{jk}{}^i + S_{kj}{}^i$$

in the definition of angular velocity.

In the general case of affine motion the formula (3.32) will be written in any of the abbreviated forms

$$\widehat{\Omega} = \widehat{\Omega}_{\rm dr} + \widehat{\Omega}_{\rm rl} = \widehat{\Omega}({\rm dr}) + \widehat{\Omega}({\rm rl}),$$

where

$$\widehat{\Omega}_{\rm dr}{}^{A}{}_{B} = \widehat{\Omega}({\rm dr})^{\rm A}{}_{\rm B} = \varphi^{-1A}{}_{F}\Gamma^{F}{}_{DC}\varphi^{D}{}_{B}\varphi^{C}{}_{E}\widehat{V}^{E}, \tag{3.38}$$

$$\widehat{\Omega}_{\rm rl}{}^{A}{}_{B} = \widehat{\Omega}({\rm rl})^{\rm A}{}_{\rm B} = \varphi^{-1A}{}_{C}\frac{d\varphi^{C}{}_{B}}{dt}. \tag{3.39}$$

The labels “dr” and “rl” refer respectively to “drift” (or ”drive”) and “relative”. The reason is that, as mentioned, φ refers to affine rotations with respect to the just passed prescribed reference frame E; the first term describes the time rate of affine rotations contained in the field E itself. When gyroscopic constraints are imposed, all these $\widehat{\Omega}$-objects become skew-symmetric angular velocities. To stress this, sometimes, but not always, we shall use then the symbols $\widehat{\omega}$, $\widehat{\omega}_{\rm dr} = \widehat{\omega}({\rm dr})$ and $\widehat{\omega}_{\rm rl} = \widehat{\omega}({\rm rl})$.

One of analytical advantages following from the prescribed reference frame E is the possibility of using the polar and two-polar decompositions [49],

$$\varphi = UA = BU = LDR^{-1},$$

where U, L, R are orthogonal (more precisely, η-orthogonal when artificial non-orthonormal coordinates are used in $\mathbb{R}^n$), A, B are symmetric (η-symmetric), D is diagonal, and obviously,

$$B = UAU^{-1}.$$

As usual, U, L, R denote fictitious gyroscopic degrees of freedom extracted from $\varphi \in {\rm GL}(n, \mathbb{R})$ [49]. The corresponding “co-moving” angular velocities are given by the expressions

$$\widehat{\omega}_{\rm rl} = U^{-1}\frac{dU}{dt}, \qquad \widehat{\chi}_{\rm rl} = L^{-1}\frac{dL}{dt}, \qquad \widehat{\vartheta}_{\rm rl} = R^{-1}\frac{dR}{dt}. \tag{3.40}$$

Obviously, as usual, the “spatial” representation may be used:

$$\omega_{\rm rl} = \frac{dU}{dt}U^{-1}, \qquad \chi_{\rm rl} = \frac{dL}{dt}L^{-1}, \qquad \vartheta_{\rm rl} = \frac{dR}{dt}R^{-1}.$$

However, in calculations appearing in practical problems the “co-moving” objects are more convenient. Obviously, in the two-dimensional world, when $n = 2$, these representations coincide.

After some calculations one can show that the kinetic energy of internal motion $T_{\rm int}$ (3.1) may be expressed in the following way in terms of the polar decomposition:

$$T_{\rm int} = -\frac{1}{2}{\rm Tr}\left(AJA\widehat{\omega}^2\right) + {\rm Tr}\left(AJ\frac{dA}{dt}\widehat{\omega}\right) + \frac{1}{2}{\rm Tr}\left(J\left(\frac{dA}{dt}\right)^2\right), \tag{3.41}$$

where

$$\widehat{\omega} = \widehat{\omega}_{\rm dr} + \widehat{\omega}_{\rm rl} = \widehat{\omega}_{\rm dr} + U^{-1}\frac{dU}{dt}, \tag{3.42}$$

and obviously $\widehat{\omega}_{\rm dr}$ is the restriction of $\widehat{\Omega}_{\rm dr}$ (3.38) to the U-rigid motion:

$$\widehat{\omega}_{\rm dr}{}^{A}{}_{B} = U^{-1A}{}_{F}\Gamma^{F}{}_{DC}U^{D}{}_{B}U^{C}{}_{E}\widehat{V}^{E}. \tag{3.43}$$

The formula (3.41) is written in the standard orthonormal coordinates in $\mathbb{R}^n$. Otherwise, when η_{AB} is admitted to be different than δ_{AB}, we have to replace the symbol J in (3.41) by J_η, where

$$J_\eta{}^{A}{}_{B} := J^{AC}\eta_{CB}.$$

Geometrically, J_η is a mixed tensor in $\mathbb{R}^n$, i.e., linear endomorphism of $\mathbb{R}^n$, $J_\eta \in {\rm L}(n,\mathbb{R})$, whereas J itself is a twice contravariant tensor.

The two-polar decomposition becomes analytically useful in doubly-isotropic dynamical problems, i.e., ones isotropic both in the physical space M and the micromaterial space. This double isotropy imposes certain restrictions both on the kinetic and potential energies. What concerns the very kinetic energy, the inertial tensor must be proportional to $\widetilde{\eta}$, i.e.,

$$J^{AB} = I\eta^{AB}, \qquad J_\eta{}^{A}{}_{B} = I\delta^{A}{}_{B}.$$

Then one can show that (3.41) becomes

$$T_{\rm int} = -\frac{I}{2}{\rm Tr}\left(D^2\widehat{\chi}^2\right) - \frac{I}{2}{\rm Tr}\left(D^2\widehat{\vartheta}^2\right) + I{\rm Tr}\left(D\widehat{\chi}D\widehat{\vartheta}\right), \tag{3.44}$$

where now

$$\widehat{\vartheta} = R^{-1}\frac{dR}{dt}, \tag{3.45}$$

$$\widehat{\chi} = \widehat{\chi}_{\rm dr} + \widehat{\chi}_{\rm rl} = \widehat{\chi}_{\rm dr} + L^{-1}\frac{dL}{dt}, \tag{3.46}$$

$$\widehat{\chi}_{\rm dr}{}^{A}{}_{B} = L^{-1A}{}_{F}\Gamma^{F}{}_{DC}L^{D}{}_{B}L^{C}{}_{E}\widehat{V}^{E}. \tag{3.47}$$

The last formula is quite analogous to (3.43). Just like there, $\widehat{\chi}$ contains the "drive" term built of the connection coefficients. It is only the L-rotation that is coupled in this way to spatial geometry; the R-rotation is geometry-independent.

Again we conclude that (3.44) is structurally identical with the corresponding formula for extended affine bodies [37] with the proviso however that $\widehat{\chi}$ contains the drive-term. The expression for $\widehat{\vartheta}$ is free of such a correction. Everything that has to do with (M, Γ, g)-geometry is absorbed by the $\widehat{\chi}$-term.

Chapter 4

Special two-dimensional problems

We shall consider now some special two-dimensional cases, i.e., when $n = 2$. Therefore, for the infinitesimal rigid body (infinitesimal gyroscope) we are dealing with three degrees of freedom: two translational and one internal, rotational. If no gyroscopic constraints are imposed and the internal motion is affine, then of course there are four internal degrees of freedom; together with translational motion one obtains six degrees of freedom. The resulting models are interesting in themselves from the point of view of pure analytical mechanics, in particular, some integrability and hyperintegrability (degeneracy) problems may be effectively studied. Obviously, the explicit analytical results exist only in Riemann manifolds (M, g) with some peculiar structure, first of all (but not only) in constant-curvature spaces. Some practical applications of two-dimensional models also seem to be possible, e.g., in geophysical problems, in mechanics of structured micropolar and micromorphic shells, etc. What concerns geophysics, we mean, e.g., motion of continental plates. Motion of pollutions like oil spots on the oceanic surface is another suggestive example.

Below we consider in some details three kinds of two-dimensional problems, namely motion of structured material points on the sphere, pseudo-sphere (Lobatchevski space), and torus manifolds with geometry induced by injections in the three-dimensional Euclidean space $\mathbb{R}^3$. Many interesting dynamical models, including some quite realistic ones, may be effectively investigated in analytical terms. It is not very surprising in spherical and pseudo-spherical geometries because of exceptional properties of constant-curvature spaces. But there exist also nice completely integrable models on the $\mathbb{R}^3$-injected torus. Perhaps this may have something to do with that that all these manifolds are algebraic ones (of the second degree in the spherical and pseudo-spherical cases, and of the fourth degree in toroidal geometry).

4.1 Spherical case

Let us begin with the spherical geometry. The two-dimensional "world" will be realized as the two-dimensional sphere in $\mathbb{R}^3$ with the radius R and the centre at the beginning of coordinates. Of course, the radius has the intrinsic sense, namely the scalar curvature equals $2/R^2$.

We introduce the "polar" coordinates (r, φ). Obviously, r is the geodetic distance measured from the "North Pole", i.e., $\vartheta = r/R$ is a modified "geographic latitude", and φ is "geographic longitude", i.e., $R\varphi$ is the distance measured along "equator". So, φ, ϑ are usual polar angles in $\mathbb{R}^3$, and R is the fixed radius-distance from the origin. Euclidean coordinates of points on $S^2(0, R) \in \mathbb{R}^3$ are given by the usual expressions:

$$x = R \sin\vartheta \cos\varphi, \quad y = R \sin\vartheta \sin\varphi, \quad z = R\cos\vartheta, \quad x^2 + y^2 + z^2 = R^2.$$

Obviously, r runs over the range $[0, \pi R]$ from the "North Pole" to the "South Pole", and φ has the usual range $[0, 2\pi]$ of the polar angle. Coordinates are singular at $r = 0$, $r = \pi R$, and there is the obvious 0, 2π ambiguity of the longitude φ.

Restricting the metric tensor of $\mathbb{R}^3$ (roughly speaking, the metric element $dx^2+dy^2+dz^2$) to $S^2(0, R)$ we obtain the obvious expression for the arc element on the sphere

$$ds^2 = dr^2 + R^2 \sin^2 \frac{r}{R}\, d\varphi^2. \tag{4.1}$$

So, the kinetic energy of translational motion is given by

$$T_{\mathrm{tr}} = \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + R^2 \sin^2 \frac{r}{R}\left(\frac{d\varphi}{dt}\right)^2\right), \tag{4.2}$$

where m denotes the mass of the material point.

Even for the purely translational motion some interesting questions arise, e.g., what are spherically symmetric potentials $V(r)$ for which all orbits are closed? Obviously we mean problems based on Lagrangians

$$L_{\mathrm{tr}} = T_{\mathrm{tr}} - V(r).$$

This is a counterpart of the famous Bertrand problem in $\mathbb{R}^2$. And it may be shown that the answer is similar [20, 35, 40], i.e., the possible potentials are as follows:

(*i*) **oscillatory potentials:**

$$V(r) = \frac{\varkappa}{2} R^2 \mathrm{tg}^2 \frac{r}{R}, \tag{4.3}$$

(*ii*) **Kepler-Coulomb potentials:**

$$V(r) = -\frac{\alpha}{R} \mathrm{ctg} \frac{r}{R}. \tag{4.4}$$

Obviously, with the spherical topology also the geodetic problem belongs here:

(*iii*) $V(r) = 0$, i.e., (in a sense) the special case of (*i*) or (*ii*) when $\varkappa = 0$, $\alpha = 0$.

There is an obvious correspondence with the flat-space Bertrand problem; it is suggested by the very asymptotics for $r \approx 0$, i.e.,

$$V(r) \approx \frac{\varkappa}{2} r^2, \qquad V(r) \approx -\frac{\alpha}{r}.$$

Obviously, this is a rough argument, but it may be shown [35, 40] that there exists a rigorous isomorphism based on the projective geometry.

The mentioned Bertrand models lead to completely integrable and maximally degenerate (hyperintegrable) problems. But even for the simplest, i.e., geodetic, models with the internal degrees of freedom the situation drastically changes. There exist interesting and practically applicable integrable models, but as a rule interaction with internal degrees of freedom reduces or completely removes degeneracy.

Let us begin with the gyroscopic model of internal motion. Unlike the general case, in two-dimensional problems with the constant-curvature spaces, gyroscopic problem is simpler than affine one. Moreover, it very simplifies study of the affine case.

The first step is to introduce an appropriate field E, i.e., the auxiliary and fixed once for all orthonormal aholonomic reference frame. This is often the matter of inventive guessing. In this case it is natural to expect that the natural base (holonomic one) tangent to coordinate lines is a good starting point. It consists of the vector fields

$$\mathcal{E}_r = \frac{\partial}{\partial r}, \qquad \mathcal{E}_\varphi = \frac{\partial}{\partial \varphi}.$$

Obviously, we mean here the well-known identification between vector fields and first-order differential operators, i.e.,

$$X = X^i \frac{\partial}{\partial x^i}$$

(X^i are components with respect to the manifold coordinates x^i).

Using more traditional language: if coordinates are ordered as (r, φ), then $\mathcal{E}_r$ and $\mathcal{E}_\varphi$ have respectively components $[1,0]^T$ and $[0,1]^T$. This holonomic system is not appropriate however, because it is not orthonormal (no holonomic system may be so in a curved Riemann space). However, it is evidently orthogonal (r, φ are orthogonal coordinates):

$$g_{r\varphi} = g_{\varphi r} = 0, \qquad g_{rr} = 1, \qquad g_{\varphi\varphi} = R^2 \sin^2 \frac{r}{R}.$$

The lengths of $\mathcal{E}_r$, $\mathcal{E}_\varphi$ are obviously given by $\|\mathcal{E}_r\| = 1$, $\|\mathcal{E}_\varphi\| = R \sin(r/R)$. So, it is natural to expect that the normalized fields

$$E_r = \frac{\partial}{\partial r} = \mathcal{E}_r, \qquad E_\varphi = \frac{1}{R \sin \frac{r}{R}} \frac{\partial}{\partial \varphi} = \frac{1}{R \sin \frac{r}{R}} \mathcal{E}_\varphi \tag{4.5}$$

will form a convenient orthonormal frame E. Let us stress that E is aholonomic, although directions of E_r, E_φ are tangent respectively to coordinate lines of r, φ. The reason is just the "crossed-variables" normalization, i.e., $\mathcal{E}_\varphi$ is multiplied by a function dependent on r. There are no coordinates r', φ' for which we would have $E_r = \partial/\partial r'$, $E_\varphi = \partial/\partial \varphi'$. The orthonormal frame $e = (e_1, e_2)$ describing the internal configuration is obtained from the fixed aholonomic frame $E = (E_r, E_\varphi)$ with the help of some time-dependent orthogonal matrix $\varphi = U$, i.e.,

$$\begin{aligned} e_1 &= E_1 U^1{}_1 + E_2 U^2{}_1 = E_r U^r{}_1 + E_\varphi U^\varphi{}_1, \\ e_2 &= E_1 U^1{}_2 + E_2 U^2{}_2 = E_r U^r{}_2 + E_\varphi U^\varphi{}_2. \end{aligned}$$

Obviously, U may be parameterized in the usual way, i.e.,

$$U = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}$$

and

$$\widehat{\omega}_{\mathrm{rl}} = \omega_{\mathrm{rl}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \frac{d\psi}{dt} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Remark: do not confuse the scalar quantity ω_{rl} with the second order skew-symmetric tensor $\widehat{\omega}_{\mathrm{rl}}$ parameterized by it.

Similarly we write that

$$\widehat{\omega}_{\mathrm{dr}} = \omega_{\mathrm{dr}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \widehat{\omega} = \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

where

$$\begin{aligned}\omega_{\rm dr} &= \cos\frac{r}{R}\frac{d\varphi}{dt} = \cos\vartheta\frac{d\varphi}{dt},\\ \omega &= \omega_{\rm rl} + \omega_{\rm dr} = \frac{d\psi}{dt} + \cos\frac{r}{R}\frac{d\varphi}{dt} = \frac{d\psi}{dt} + \cos\vartheta\frac{d\varphi}{dt}.\end{aligned} \qquad (4.6)$$

Obviously, the above expressions are the special cases of (3.40), (3.42), (3.43) and follow easily from the formulas for Christoffel symbols (Levi-Civita connection) on the sphere, i.e.,

$$\Gamma^{r}{}_{\varphi\varphi} = -\frac{R}{2}\sin\frac{r}{R}, \qquad \Gamma^{\varphi}{}_{r\varphi} = \Gamma^{\varphi}{}_{\varphi r} = \frac{1}{R}\mathrm{ctg}\frac{r}{R}$$

(the remaining coefficients vanish). In the aholonomic representation:

$$\Gamma^{r}{}_{\varphi\varphi} = -\frac{1}{R}\mathrm{ctg}\frac{r}{R}, \qquad \Gamma^{\varphi}{}_{r\varphi} = \frac{1}{R}\mathrm{ctg}\frac{r}{R},$$

but let us notice that

$$\Gamma^{\varphi}{}_{\varphi r} = 0 \neq \Gamma^{\varphi}{}_{r\varphi},$$

i.e., the aholonomic coefficients are in general not symmetric. The remaining aholonomic coefficients also vanish. Obviously, the above $\Gamma^{a}{}_{bc}$ are exactly the formerly used $\Gamma^{A}{}_{BC}$; simply in some particular concrete cases this way of writing is more suggestive and intuitive.

After simple calculations based on the formula (3.1) or rather on its restriction to the gyroscopic motion, one obtains the expected expression:

$$\begin{aligned}T &= T_{\rm tr} + T_{\rm int} \\ &= \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + R^2\sin^2\frac{r}{R}\left(\frac{d\varphi}{dt}\right)^2\right) + \frac{I}{2}\left(\frac{d\psi}{dt} + \cos\frac{r}{R}\frac{d\varphi}{dt}\right)^2,\end{aligned} \qquad (4.7)$$

i.e., briefly,

$$T = T_{\rm tr} + T_{\rm int} = \frac{mv^2}{2} + \frac{I\omega^2}{2},$$

where ω is given by (4.6). The scalar quantity I, i.e., the inertial moments of the plane rotator is related to the tensor J by the formula

$$I = \eta_{AB}J^{AB} = \mathrm{Tr}J_{\eta},$$

i.e., in Euclidean material coordinates ($\eta_{AB} = \delta_{AB}$) simply $I = \mathrm{Tr}J$. In the absence of deformations the internal inertia is controlled only by this single scalar. This is the peculiarity of the "two-dimensional world".

For certain reasons it will be convenient to rewrite the formula (4.7) in terms of the variable $\vartheta = r/R$, i.e.,

$$\begin{aligned} T &= T_{\rm tr} + T_{\rm int} \\ &= \frac{mR^2}{2}\left(\left(\frac{d\vartheta}{dt}\right)^2 + \sin^2\vartheta\left(\frac{d\varphi}{dt}\right)^2\right) + \frac{I}{2}\left(\frac{d\psi}{dt} + \cos\vartheta\frac{d\varphi}{dt}\right)^2. \end{aligned} \tag{4.8}$$

It is seen that if formally $(\varphi, \vartheta, \psi)$ are interpreted as Euler angles (respectively the precession, nutation, and rotation), the above expression is formally identical with the kinetic energy of the three-dimensional symmetric rigid body (without translations) with the main moments of inertia given respectively by

$$I_1 = I_2 = mR^2, \qquad I_3 = I.$$

If $I = mR^2$ one obtain the expression for the spherical top.

There is nothing surprising in the mentioned isomorphism because the quotient manifold $\mathrm{SO}(3,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ may be in a natural way identified with $\mathrm{S}^2(0,1)$ (or with any $\mathrm{S}^2(0,R)$). Projecting the motion of the three-dimensional symmetric top onto the quotient sphere-manifold we obtain two-dimensional translational motion; the one-dimensional subgroup of rotations about the z-axis refers to the internal motion of the two-dimensional rotator.

The projection procedure is exactly compatible with the mentioned correspondence between Euler angles in $\mathrm{SO}(3,\mathbb{R})$ and our generalized coordinates φ, $\vartheta = r/R$, ψ of the infinitesimal rotator in $\mathrm{S}^2(0,R)$.

Let $U(\varphi,\vartheta,\psi) \in \mathrm{SO}(3,R)$ be just the element labelled by the Euler angles φ, ϑ, ψ, thus

$$U(\varphi,\vartheta,\psi) = U_z(\varphi)U_x(\vartheta)U_z(\psi), \tag{4.9}$$

where U_z, U_x are rotations respectively around the z- and x-axes; angles of rotations are indicated as arguments. Calculating the "co-moving angular velocity"

$$\widehat{\varkappa} = U^{-1}\frac{dU}{dt} \tag{4.10}$$

of this fictitious three-dimensional top one obtains that

$$\widehat{\varkappa} = \widehat{\varkappa}_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + \widehat{\varkappa}_2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \widehat{\varkappa}_3 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.11}$$

where

$$\widehat{\varkappa}_1 = \sin\vartheta\sin\psi\frac{d\varphi}{dt} + \cos\psi\frac{d\vartheta}{dt}, \tag{4.12}$$

$$\widehat{\varkappa}_2 = \sin\vartheta\cos\psi\frac{d\varphi}{dt} - \sin\psi\frac{d\vartheta}{dt}, \tag{4.13}$$

$$\widehat{\varkappa}_3 = \cos\vartheta\frac{d\varphi}{dt} + \frac{d\psi}{dt}. \tag{4.14}$$

In expression (4.14) we easily recognize (4.6), i.e., the expression for the one-component angular velocity of the two-dimensional rotator. Calculating formally the kinetic energy of the three-dimensional symmetric $\mathrm{SO}(3,\mathbb{R})$-top, i.e.,

$$T = \frac{K}{2}\left(\widehat{\varkappa}_1\right)^2 + \frac{K}{2}\left(\widehat{\varkappa}_2\right)^2 + \frac{I}{2}\left(\widehat{\varkappa}_3\right)^2, \tag{4.15}$$

and substituting $K = mR^2$, $\vartheta = r/R$, we obtain exactly (4.7), i.e., (4.8).

The principal fibre bundle of orthonormal frames $\mathrm{F}(\mathrm{S}^2, g)$ over the two-dimensional sphere (with its induced metric g) has $\mathrm{SO}(2,\mathbb{R})$ as the structural group and it may be itself identified with $\mathrm{SO}(3,\mathbb{R})$. This is just the essence of the above identification between $\mathrm{SO}(3,\mathbb{R})$ and its quotient manifold $\mathrm{S}^2 \simeq \mathrm{SO}(3,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$. Some important invariance problems appear there. Namely, as usual in analytical mechanics, the kinetic energy (4.7), (4.8) may be identified with some Riemannian structure on the configuration space. Let us write down our kinetic energy in the following form with the explicitly separated mass factor:

$$T = \frac{m}{2}G_{ij}(q)\frac{dq^i}{dt}\frac{dq^j}{dt}.$$

Just as above, our generalized coordinates q^i, $i = 1, 2, 3$, are the variables (r, φ, ψ) written just in this direction. On the level of $\mathrm{SO}(3,\mathbb{R})$ as identified with $\mathrm{F}(\mathrm{S}^2, g)$, they are equivalent to the Euler angles $(\vartheta, \varphi, \psi)$, where $\vartheta = r/R$.

After some calculations one obtains that

$$[G_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2\sin^2\frac{r}{R} + \frac{I}{m}\cos^2\frac{r}{R} & \frac{I}{m}\cos\frac{r}{R} \\ 0 & \frac{I}{m}\cos\frac{r}{R} & \frac{I}{m} \end{bmatrix}.$$

In the special case $I = mR^2$ one obtains that G simplifies to $\breve{G}$, where

$$\left[\breve{G}_{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & R^2\cos\frac{r}{R} \\ 0 & R^2\cos\frac{r}{R} & R^2 \end{bmatrix}. \tag{4.16}$$

The corresponding expressions for the weight-one volume densities are as follows:

$$\sqrt{G} = R\sqrt{\frac{I}{m}}\sin\frac{r}{R}, \qquad \sqrt{\breve{G}} = R^2\sin\frac{r}{R}.$$

The contravariant inverse metric G^{ij} $(G^{ik}G_{kj} = \delta^i{}_j)$ is given as follows:

$$\left[G^{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{R^2\sin^2\frac{r}{R}} & -\frac{\cos\frac{r}{R}}{R^2\sin^2\frac{r}{R}} \\ 0 & -\frac{\cos\frac{r}{R}}{R^2\sin^2\frac{r}{R}} & \frac{m}{I} + \frac{1}{R^2}\mathrm{ctg}^2\frac{r}{R} \end{bmatrix},$$

and, obviously,

$$\left[\breve{G}^{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{R^2\sin^2\frac{r}{R}} & -\frac{\cos\frac{r}{R}}{R^2\sin^2\frac{r}{R}} \\ 0 & -\frac{\cos\frac{r}{R}}{R^2\sin^2\frac{r}{R}} & \frac{1}{R^2\sin^2\frac{r}{R}} \end{bmatrix}.$$

If we once identify the bundle manifold $F(S^2, g)$ with the group $\mathrm{SO}(3,\mathbb{R})$, then we can consider the action of two transformation groups. They are, obviously, represented by the left and right regular actions of $\mathrm{SO}(3,\mathbb{R})$ on itself, i.e.,

$$X \mapsto VX, \qquad X \mapsto XV,$$

where $X, V \in \mathrm{SO}(3,\mathbb{R})$. From the point of view of the original configuration space $F(S^2, g)$, these groups act both on the positions of material point in S^2 and orientations of the attached bases, i.e., internal configurations.

The above metrics G are invariant under the total left-acting group $\mathrm{SO}(3,\mathbb{R})$. What concerns the right actions, in general they are invariant only under the subgroup $\mathrm{SO}(2,\mathbb{R}) \subset \mathrm{SO}(3,\mathbb{R})$ interpreted as the group of rotations about the z-axis in $\mathbb{R}^3$. Therefore, G has the four-dimensional isometry group $\mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(2,\mathbb{R})$. Only in the special case $I = mR^2$ the metric tensor G is invariant under $\mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(3,\mathbb{R})$, i.e., under all left and right regular translations. Therefore, on the level of $\mathrm{SO}(3,\mathbb{R})$-description we are dealing then with the spherical top and up to a constant multiplier G becomes the Killing metric on $\mathrm{SO}(3,\mathbb{R})$. Of course, this special case is not interesting from the point of view of our primary model for an infinitesimal rigid body moving in $S^2(0,\mathbb{R})$ because it is there physically too exotic and too exceptional.

For the potential systems with Lagrangians $L = T - V(q)$ the Legendre transformation has the usual form:

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = mG_{ij}(q)\dot{q}^j.$$

Inverting it,

$$\dot{q}^i = \frac{1}{m}G^{ij}(q)p_j, \tag{4.17}$$

and substituting (4.17) to the expression for energy,

$$E = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L = T + V(q),$$

one obtains the Hamiltonian H, i.e.,

$$H(q,p) = \mathcal{T}(q,p) + V(q) = \frac{1}{2m}G^{ij}(q)p_i p_j + V(q),$$

in particular, the geodetic Hamiltonian (when $V(q) = 0$):

$$\mathcal{T} = \frac{1}{2m}G^{ij}(q)p_i p_j.$$

As a rule, the presence of forces, i.e., not constant $V(q)$, reduces strongly the aforementioned symmetry of geodetic models.

We are interested here in integrability and hyperintegrability (degeneracy) problems, thus, we concentrate our attention on the Hamilton-Jacobi equation, i.e.,

$$\frac{\partial S}{\partial t} + H\left(q^i, \frac{\partial S}{\partial q^i}\right) = 0,$$

its separability, and the action-angle variables. We are dealing only with time-independent problems, thus, $S(t,q)$ is always sought in the following form:

$$S(t,q) = -Et + S_0(q),$$

where, obviously, the reduced action S_0 satisfies the time-independent Hamilton-Jacobi equation, i.e.,

$$H\left(q^i, \frac{\partial S_0}{\partial q^i}\right) = E. \tag{4.18}$$

Let us write down explicitly a few formulas describing Legendre transformation and kinetic energy for the rigid body moving in two-dimensional spherical world.

Denoting canonical momenta conjugate to coordinates (r, φ, ψ) respectively by (p_r, p_φ, p_ψ), we have that

$$\begin{aligned} p_r &= m\dot{r}, \\ p_\varphi &= mR^2 \left(\sin^2 \frac{r}{R} + \frac{I}{mR^2} \cos^2 \frac{r}{R} \right) \dot{\varphi} + I \cos \frac{r}{R} \dot{\psi}, \\ p_\psi &= I\dot{\psi} + I \cos \frac{r}{R} \dot{\varphi}. \end{aligned}$$

The resulting geodetic Hamiltonian has the following form:

$$\begin{aligned} \mathcal{T} &= \frac{p_r^2}{2m} + \frac{p_\varphi^2 - 2p_\varphi p_\psi \cos \frac{r}{R} + \left(\frac{mR^2}{I} \sin^2 \frac{r}{R} + \cos^2 \frac{r}{R} \right) p_\psi^2}{2mR^2 \sin^2 \frac{r}{R}} \\ &= \frac{p_r^2}{2m} + \frac{p_\varphi^2 - 2p_\varphi p_\psi \cos \frac{r}{R} + p_\psi^2}{2mR^2 \sin^2 \frac{r}{R}} + \frac{mR^2 - I}{2mR^2 I} p_\psi^2. \end{aligned}$$

It is seen again how this expression simplifies in the special case when $I = mR^2$.

Without gyroscopic degree of freedom, when T is reduced to T_{tr} given by the formula (4.2), $\mathcal{T}$ is reduced to

$$\mathcal{T}_{\text{tr}} = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mR^2 \sin^2 \frac{r}{R}}.$$

Then there exists the class of separable potentials

$$V(r, \varphi) = V_r(r) + \frac{V_\varphi(\varphi)}{R^2 \sin^2 \frac{r}{R}}.$$

With rotational degree of freedom the metric tensor G is not diagonal in natural coordinates (r, φ, ψ) and there is no direct analogue of above integrable models. Of course, in three-dimensional Riemann manifolds there are always orthogonal coordinates, but it is fairly not obvious whether there exist diagonalizing coordinates admitting a physically reasonable class of potentials treatable in terms of the separation-of-variables method.

It is seen however that φ, ψ are cyclic variables in the kinetic energy term; this focuses our attention on the models where the potential energy also does not depend on φ, ψ. These angles are then cyclic variables for the total Lagrangian $L = T - V$ and the corresponding conjugate momenta p_φ, p_ψ are constants of motion. The resulting models, including geodetic ones ($V = 0$), are completely integrable and one can analyze them by means of the separation-of-variables method.

As usual when dealing with cyclic variables, we seek the reduced action S_0 in the following form:

$$S_0(r,\varphi,\psi;E,\ell,s) = S_r(r;E) + \ell\varphi + s\psi, \tag{4.19}$$

where ℓ, s are integration constants, i.e., the dependence of $S_\varphi(\varphi)$ and $S_\psi(\psi)$ on their arguments is postulated as linear. Together we have three integration constants E,ℓ,s, just as it should be in a complete integral for the system with three degrees of freedom. As a matter of fact, due to the assumed symmetry, the problem reduces to the one-dimensional one for S_r and substituting (4.19) into (4.18), we obtain the ordinary differential equation

$$\begin{aligned}\left(\frac{dS_r}{dr}\right)^2 &= 2m\,(E - V(r)) \\ &\quad - \frac{\ell^2 - 2\ell s\cos\frac{r}{R} + s^2\left(\frac{mR^2}{I}\sin^2\frac{r}{R} + \cos^2\frac{r}{R}\right)}{R^2\sin^2\frac{r}{R}}.\end{aligned} \tag{4.20}$$

Therefore, $p_r = dS_r/dr$ equals plus-minus (depending on the phase of motion) square root of the rigid-hand side of the expression above. Obviously, this is well defined only in the classically admissible region between the turning points. It is clear that

$$p_\varphi = \frac{\partial S_0}{\partial\varphi} = \frac{dS_\varphi}{d\varphi} = \ell, \qquad p_\psi = \frac{\partial S_0}{\partial\psi} = \frac{dS_\psi}{d\psi} = s.$$

The corresponding action variables are given as follows:

$$\begin{aligned} J_\varphi &= \oint p_\varphi d\varphi = \int_0^{2\pi} \ell d\varphi = 2\pi\ell, \\ J_\psi &= \oint p_\psi d\psi = \int_0^{2\pi} s d\psi = 2\pi s. \end{aligned}$$

The radial action variable

$$J_r = \oint p_r dr$$

equals the doubled integral of the square-rooted right-hand side of (4.20) between the turning points, i.e., between nulls of (4.20). Substituting there

$$\ell = \frac{J_\varphi}{2\pi}, \qquad s = \frac{J_\psi}{2\pi}, \tag{4.21}$$

we obtain the expression

$$J_r = \oint \sqrt{2m\left(E - V(r)\right) - \frac{\left(J_\varphi - J_\psi \cos\frac{r}{R}\right)^2}{4\pi^2 R^2 \sin^2\frac{r}{R}} + \frac{mJ_\psi^2}{4\pi^2 I}}\, dr. \tag{4.22}$$

Let us remind that the above expression, i.e., the contour integral of the differential one-form $p_r dr$ along the corresponding orbit in the two-dimensional phase space of the (r, p_r)-variables, equals

$$J_r = 2\int_{r_{\min}}^{r_{\max}} p_r(r)dr,$$

where $r_{\min}, r_{\max}$ denote respectively the left and right turning points of the r-motion.

When some explicit form of $V(r)$ is assumed and substituted to (4.22), then in principle the integral may be calculated and one obtains the functional dependence of the quantity J_r on the integration constants E, J_φ, J_ψ, i.e.,

$$J_r = J_r\left(E, J_\varphi, J_\psi\right).$$

Once calculated, this expression may be in principle solved with respect to the energy E, i.e.,

$$E = \mathcal{H}\left(J_r, J_\varphi, J_\psi\right).$$

In this way the energy is expressed in terms of action variables. Substituting this expression and (4.21) into (4.19), we obtain the generating function of canonical transformation from the original phase-space coordinates $(r, \varphi, \psi; p_r, p_\varphi, p_\psi)$ to the action-angle variables $(\Theta_r, \Theta_\varphi, \Theta_\psi; J_r, J_\varphi, J_\psi)$, i.e.,

$$S_0\left(r, \varphi, \psi; J_r, J_\varphi, J_\psi\right) = S_r\left(r, \mathcal{H}\left(J_r, J_\varphi, J_\psi\right)\right) + \frac{\varphi}{2\pi}J_\varphi + \frac{\psi}{2\pi}J_\psi.$$

The resulting angle quantities are given as follows:

$$\Theta_r = \frac{\partial S_0}{\partial J_r}, \qquad \Theta_\varphi = \frac{\partial S_0}{\partial J_\varphi}, \qquad \Theta_\psi = \frac{\partial S_0}{\partial J_\psi}.$$

This “angles” are meant modulo 1 but perhaps more intuitive are angles taken modulo 2π, i.e.,

$$\widetilde{\Theta}_r = 2\pi\Theta_r, \qquad \widetilde{\Theta}_\varphi = 2\pi\Theta_\varphi, \qquad \widetilde{\Theta}_\psi = 2\pi\Theta_\psi.$$

Let us observe that usually

$$\widetilde{\Theta}_{\varphi} \neq \varphi, \qquad \widetilde{\Theta}_{\psi} \neq \psi.$$

The fundamental frequencies are given as follows:

$$\nu_r = \frac{\partial \mathcal{H}}{\partial J_r}, \qquad \nu_{\varphi} = \frac{\partial \mathcal{H}}{\partial J_{\varphi}}, \qquad \nu_{\psi} = \frac{\partial \mathcal{H}}{\partial J_{\psi}},$$

and their circular counterparts are the corresponding 2π-multipliers, i.e.,

$$\omega_r = 2\pi\nu_r, \qquad \omega_{\varphi} = 2\pi\nu_{\varphi}, \qquad \omega_{\psi} = 2\pi\nu_{\psi}.$$

As usual, the action variables are constants of motion, i.e.,

$$\frac{dJ_r}{dt} = \frac{\partial \mathcal{H}}{\partial \Theta_r} = 0, \qquad \frac{dJ_{\varphi}}{dt} = \frac{\partial \mathcal{H}}{\partial \Theta_{\varphi}} = 0, \qquad \frac{dJ_{\psi}}{dt} = \frac{\partial \mathcal{H}}{\partial \Theta_{\psi}} = 0.$$

And as usual, the angular variables perform uniform "librations" because

$$\frac{d\Theta^i}{dt} = \frac{\partial \mathcal{H}}{\partial J_i} = \nu^i(J) = \text{const},$$

thus,

$$\Theta^i(t) = \nu^i(J)t + \alpha^i, \qquad \widetilde{\Theta}^i(t) = \omega^i(J)t + \beta^i,$$

and, obviously, the index i runs over the labels r, φ, ψ.

A very important point is hyperintegrability, i.e., degeneracy. Let us remind that a multiply periodic system with n degrees of freedom (an integrable one with an open set of bounded trajectories) is said to be k-fold degenerate (or $(n-k)$-fold periodic) when fundamental frequencies satisfy a system of k (but not more) independent equations, i.e.,

$$n^{\alpha}{}_{i}\nu^i(J) = 0, \qquad \alpha = \overline{1, k},$$

where $n^{\alpha}{}_{i}$ are integers, i.e., $n^{\alpha}{}_{i} \in \mathbb{Z}$ (but nothing changes when we say that they are rationals, i.e., $n^{\alpha}{}_{i} \in \mathbb{Q}$). By "independent equations" we mean obviously that $\text{Rank}\,[n^{\alpha}{}_{i}] = k$. Integers $n^{\alpha}{}_{i}$ are to be constant, independent of J, thus, the same for all possible frequencies. If the above equations hold exceptionally for certain exclusive values of J, one says about accidental degeneracy.

Of course the n-dimensional manifolds $J_i = \text{const}$ are tori. While $\{J_i, J_k\} = 0$, i.e., the action variables are in Poisson-bracket involution, these tori are Lagrangian submanifolds, i.e., the symplectic two-form vanishes when evaluated on any pair of their tangent vectors at any point, and they are maximal connected manifolds of this property. If the system is k-fold degenerate ($(n-k)$-fold periodic), then the closure of every trajectory is an $(n-k)$-dimensional isotropic torus (the symplectic form vanishes when evaluated at any pair of tangent vectors attached at any point). The system orbits are dense in these tori. The mentioned $(n-k)$-tori form a regular foliation (congruence) of partially disjoint submanifolds on the Lagrangian tori $J_a = \text{const}$, $a = \overline{1,n}$. For any k-fold degenerate system Hamiltonian depends on the action variables $J_1, \ldots, J_n$ in a very peculiar way, namely, it is a function of some $(n-k)$ linear combination of $J_1, \ldots, J_n$ with integer coefficients (and of no less number of such combinations). The extreme examples are as follows:

(i) if $k = 0$, then the system is completely nondegenerate, any orbit fills densely some torus $J_a = \text{const}$, $a = \overline{1,n}$, and H depends on all J_a-variables in an essential way, i.e., there is no possibility to superpose them into a smaller numbers of combinations with integer coefficients,

(ii) if $k = n-1$, then all bounded trajectories are periodic, i.e., they are topological circles (one-dimensional tori) regularly foliating the tori $J_a = \text{const}$, $a = \overline{1,n}$, so that the quotient manifolds are $(n-1)$-dimensional tori. Then there exists a combination of J_a-variables with integer coefficients, i.e., $J = n^a J_a$, $n^a \in \mathbb{Z}$, such that H depends on all J_a-s only through J.

The first special case is completely not resonant and has n independent frequencies (periods). The second, quite opposite, case is maximally resonant, i.e., there is only one fundamental frequency (period) and trajectories are known from the elementary mechanics as Lissajous figures in the original configuration space.

If the system is k-fold degenerate, then the original action-angle variables obtained from the space-time coordinates q^i, p_i, $i = \overline{1,n}$, may be replaced by some new ones, better expressing the degeneracy (hyperintegrability) structure. Namely, it is always possible to introduce some quantities $\widetilde{\Theta}^a$, $\widetilde{J}_a$ such that

(i) $J_a = \widetilde{J}_b N^b{}_a$, $\widetilde{\Theta}^a = N^a{}_b \Theta^b$,

(ii) $N^a{}_b \in \mathbb{Z}$, i.e., entries of the matrix N are integers,

(iii) $\det [N^a{}_b] = \pm 1$,

(iv) $H = \mathcal{H}\left(\widetilde{J}_1, \ldots, \widetilde{J}_{n-k}\right)$, i.e., Hamiltonian depends only on the first $(n-k)$-tuple of the action variables,

(v) the first $(n-k)$ basic frequencies $\widetilde{\nu}^r$, $r=\overline{1,(n-k)}$, are independent over integers, i.e., the equations

$$\sum_{r=1}^{n-k} n^s{}_r \widetilde{\nu}^r = \sum_{r=1}^{n-k} n^s{}_r \frac{\partial \mathcal{H}}{\partial \widetilde{J}_r}$$

imply that the $(n-k)\times(n-k)$ matrix $n^s{}_r$ vanishes, i.e., $\mathbb{Z} \ni n^s{}_r = 0$.

Obviously, the remaining k frequencies, i.e., $\widetilde{\nu}^p = \partial\mathcal{H}/\partial J_p = 0$ with $p = \overline{(n-k+1),n}$, vanish and the corresponding new action variables $\widetilde{\Theta}^p$, $p = \overline{(n-k+1),n}$, are constants of motion. Together we have $(n+k)$ constants of motion, global and smooth ones:

$$\widetilde{J}_1, \ldots, \widetilde{J}_n, \widetilde{\Theta}^{n-k+1}, \ldots, \widetilde{\Theta}^n.$$

What concerns globality, according to the above conditions (i) and (ii), the quantities $\widetilde{\Theta}^a$, $a=\overline{1,n}$, are "good" angular variables (they would not be so without these conditions). They are of course multivalued in a "harmless" way when property treated, just like the angular variables. To avoid this multivaluedness, one can replace them by unimodular complex numbers, i.e.,

$$\zeta^a = \exp\left(i\Theta^a\right).$$

Let us go back from these general digressions to our special models, beginning from (4.22), thus, to $n=3$.

It happens often, e.g., in the case of symmetry in the material point mechanics, that some partial features of the problem may be seen from the formulas like (4.22) for J_r even without calculating the integral and without even assuming anything about the shape of $V(r)$. For example, in central problems of the material point motion, when the spherical coordinates r, ϑ, φ are used, it is seen that the corresponding action variables J_ϑ, J_φ enter the under-square-root expression through the combination $J_\vartheta + J_\varphi$, so the one-fold degeneracy is obvious from the very beginning. The particular shapes of $V(r)$ are necessary for answering the question concerning total degeneracy. And then, according to what is known from somewhere else, really one can prove after the explicit integration, that the total degeneracy occurs for the Kepler-Coulumb (attractive) problem and the isotropic harmonic oscillator, i.e.,

$$V = -\frac{\alpha}{r}, \qquad V = \frac{\varkappa}{2} r^2, \qquad \alpha > 0, \qquad \varkappa > 0.$$

In the formula (4.22) nothing like this is seen, even for the very special, highly-symmetric case $I = mr^2$, i.e.,

$$J_r = \oint \sqrt{2m\left(E - V(r)\right) - \frac{J_\varphi^2 - 2J_\varphi J_\psi \cos\frac{r}{R} + J_\psi^2}{4\pi^2 R^2 \sin^2\frac{r}{R}}}dr. \tag{4.23}$$

Without performing some calculations, one does not see anything even in the geodetic case, when $V = 0$. But everybody knows that the problem is then completely degenerate because it is isomorphic, as mentioned, with the free spherical top, each orbits of which are closed (one-parameter group and their cosets). But without the explicit calculation nothing may be decided because J_φ and J_ψ do not combine integer-wise under the square root. Moreover, there are some unexpected strange technical problems when calculating then the relationship between E and J-variables. This is probably the reason one can hardly find the action-angle analysis in the rigid body mechanics. Passing to the isomorphic problem for the rigid body we obtain that

$$J_\vartheta = \oint \sqrt{2I\left(E - V(\vartheta)\right) - \frac{J_\varphi^2 - 2J_\varphi J_\psi \cos\vartheta + J_\psi^2}{4\pi^2 R^2 \sin^2\vartheta}}d\vartheta, \tag{4.24}$$

where $\vartheta = r/R$. Substituting here $x = -\cos\vartheta$, we obtain for the geodetic case that

$$J_\vartheta = -\oint \sqrt{-2IEx^2 - \frac{J_\varphi J_\psi}{2\pi^2}x + 2IE - \frac{J_\varphi^2 + J_\psi^2}{4\pi^2}}\frac{dx}{(x-1)(x+1)}. \tag{4.25}$$

This integral may be calculated according to the usual rule for integrals of the form

$$\mathcal{R}\left(\sqrt{ax^2 + bx + c}, x\right),$$

where $\mathcal{R}$ denotes a rational function of indicated expression. Of course, the corresponding indefinite integral is elementary one and can be calculated, then the resulting definite integral (the doubled value between the turning points) may be in principle obtained in this way. There is however plenty of possibilities of making mistakes and it is much more convenient to use here the method of complex integration, as elaborated by Max Born in analytical mechanics and the old quantum theory. Replacing x by the complex variable z, we see that there are exactly two branch points, i.e., just the classical turning points. Therefore, the action variable J_ϑ (J_r, J_x) may be calculated as the contour integral along the path infinitesimally surrounding the cut joining the branch point along the real axis. There are also three poles, i.e., $z = -1$, $z = +1$, $z =$

∞. Surrounding these poles by additional contours and taking the composed unconnected contour consisting of the above four ones, we finally conclude that

$$J_\vartheta = -2\pi i \mathrm{Res}_{-1} - 2\pi i \mathrm{Res}_1 - 2\pi i \mathrm{Res}_\infty,$$

where, obviously, residua are calculated for the total integrand. After some calculations one obtains that

$$\begin{aligned} \mathrm{Res}_{-1} &= -\frac{|J_\varphi - J_\psi|}{4\pi} i, \\ \mathrm{Res}_1 &= -\frac{|J_\varphi + J_\psi|}{4\pi} i, \\ \mathrm{Res}_\infty &= \sqrt{2IE} i. \end{aligned}$$

Thus, finally,

$$4\pi\sqrt{2IE} = 2J_\vartheta + |J_\varphi - J_\psi| + |J_\varphi + J_\psi| \, .$$

Only now, after all calculations, the degeneracy is seen on the level of action variables, although it was obvious from the very beginning from the spherical top analogy. There are however some delicate points, namely, the structure of this total degeneracy is a little bit different in the four regions of the phase space:

(i) $(J_\varphi > J_\psi) \wedge (J_\varphi > -J_\psi)$, i.e., $J_\varphi > |J_\psi|$, then opening the absolute value sign we obtain that

$$E = \frac{(J_\vartheta + J_\varphi)^2}{8\pi^2 I},$$

(ii) $(J_\varphi < J_\psi) \wedge (J_\varphi > -J_\psi)$, i.e., $|J_\varphi| < J_\psi$, then

$$E = \frac{(J_\vartheta + J_\psi)^2}{8\pi^2 I},$$

(iii) $(J_\varphi > J_\psi) \wedge (J_\varphi < -J_\psi)$, i.e., $|J_\varphi| < |J_\psi| = -J_\psi$, then

$$E = \frac{(J_\vartheta - J_\psi)^2}{8\pi^2 I},$$

(iv) $(J_\varphi < J_\psi) \wedge (J_\varphi < -J_\psi)$, i.e., $J_\varphi < -|J_\psi|$, then

$$E = \frac{(J_\vartheta - J_\varphi)^2}{8\pi^2 I}.$$

It is seen that there are four open regions of the phase space in which the formula looks slightly different. Nevertheless, the difference is not very essential. In any of these region one can introduce such action variables that only one of them is essential for the energy, i.e.,

$$E = \frac{J^2}{8\pi^2 I}.$$

On the level of the old quantum theory according to the Bohr-Sommerfeld postulates we have that

$$J = nh, \qquad n \in \mathbb{Z},$$

and we obtain that

$$E_n = \frac{n^2\hbar^2}{2I},$$

i.e., just as it should be.

By the way, it is seen from the formulas (4.22), (4.23), (4.24), (4.25) that the above class of integrable problems contains also interesting non-geodetic models. Obviously, this is the case when the structure of V is appropriately sited to other (geodetic) terms occurring under the square root sign. In any case, it is seen from (4.25) that potentials of the type

$$V(x) = \alpha x^2 + \beta x, \qquad \alpha, \beta = \mathrm{const}$$

do not violate the explicit solvability in terms of elementary integrals. If $\beta \neq 0$, then they also do not violate the total degeneracy because the term with the constant α will be simply absorbed by $-2IE$ and will regauge it. In other words, in (4.24) we can simply use that

$$V(\vartheta) = \widetilde{\alpha}\cos^2\vartheta + \widetilde{\beta}\cos\vartheta,$$

i.e.,

$$V(r) = \widehat{\alpha}\cos^2\frac{r}{R} + \widehat{\beta}\cos\frac{r}{R},$$

with the same consequences concerning the explicit solvability and degeneracy.

For the general case (4.22) the same is true concerning the explicit solvability, however, as a rule, the degeneracy will be removed there even for the special case $\beta = 0$ (i.e., $\widetilde{\beta} = 0$, $\widehat{\beta} = 0$).

Let us now go back to infinitesimal affinely-rigid bodies, thus, in addition to the gyroscopic degrees of freedom the deformative ones are taken into account. As mentioned, when the polar decomposition is used, then for the kinetic energy one obtains the formula (3.41) with $\widehat{\omega}$ given by the expressions (3.42) and (3.43).

Just as in the gyroscopic case we concentrate on the phenomena in the two-dimensional spherical world. We use the same as previously parametrization of this world, i.e., (r,φ)- or (ϑ,φ)-coordinates. What concerns internal degrees of freedom we use coordinates ψ, ξ, η, ζ, where

$$U(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}, \qquad A(\xi,\eta,\zeta) = \begin{bmatrix} \xi & \eta \\ \eta & \zeta \end{bmatrix},$$

and U, A are factors of the polar decomposition.

After some calculations we obtain that

$$\begin{aligned} T &= T_{\mathrm{tr}} + T_{\mathrm{int}} = \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + \sin^2\frac{r}{R}\left(\frac{d\varphi}{dt}\right)^2\right) \\ &\quad + \frac{1}{2}\left(J_1\xi^2 + (J_1+J_2)\,\eta^2 + J_2\zeta^2\right)\omega^2 \\ &\quad + \left(-J_1\eta\frac{d\xi}{dt} + (J_1\xi - J_2\eta)\frac{d\eta}{dt} + J_2\eta\frac{d\zeta}{dt}\right)\omega \\ &\quad + \frac{1}{2}\left(J_1\left(\frac{d\xi}{dt}\right)^2 + (J_1+J_2)\left(\frac{d\eta}{dt}\right)^2 + J_2\left(\frac{d\zeta}{dt}\right)^2\right), \end{aligned}$$

where the micromaterial coordinates are chosen (always possible) in such a way that the inertial quadrupole is diagonal, i.e.,

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}.$$

Obviously, ω is given by (4.6). The first term of T_{int} above, i.e., the first term of (3.41) when n is specified to 2, is the centrifugal contribution; the coefficient at $\omega^2/2$ is the dynamical deformation-dependent inertial momentum of the U-rigid body (the material tensor AJA in (3.41)). The second term is the Coriollis contribution to T_{int}, i.e., the coupling between angular and deformation velocities ($\mathrm{Tr}\,(AJ(dA/dt)\widehat{\omega})$ in (3.41)). And the third term of T_{int} is the kinetic energy of pure deformations ($\mathrm{Tr}\,\left(J(dA/dt)^2\right)/2$ in (3.41)).

It is seen that even the purely kinetic term is rather complicated and there is no hope for integrability (or any kind of rigorous solvability) even if the potential term has a simple structure adapted to that of T. Obviously, the most natural potentials are those given by scalar invariants of the Green deformation tensor, i.e., analytically

$$G = \begin{bmatrix} \xi^2+\eta^2 & \eta(\xi+\zeta) \\ \eta(\xi+\zeta) & \xi^2+\eta^2 \end{bmatrix}.$$

Obviously, in the case of some practical motivation some approximation or numerical procedures are possible, however, for the general J there is no hope for realistic potentials admitting integrability or some kind of analytical procedures.

As usual, this is possible however for highly symmetric systems, when the internal inertia is isotropic, i.e.,

$$J^{AB} = I\eta^{AB} = I\delta^{AB}.$$

This is explicitly seen when one uses the two-polar decomposition for internal degrees of freedom.

For the general n we have the expression (3.44) with $\widehat{\chi}$, $\widehat{\vartheta}$ given by the expressions (3.45), (3.46), (3.47). And there is also no hope for integrability or any kind of explicit analytical solvability. But again the special case $n = 2$, i.e., two-dimensional spherical world, is an exception.

Parametrization of the $S^2(0, R)$-world is again the same, i.e., (r, φ)- or (ϑ, φ)-variables. When expressed in terms of the two-polar decomposition, i.e.,

$$\varphi = LDR^{-1} \in \mathrm{GL}(2, \mathbb{R}),$$

then φ is parameterized by generalized coordinates $\alpha, \beta, \lambda, \mu$, where

$$L(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \qquad R(\beta) = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix},$$

$$D(\lambda, \mu) = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}.$$

In a partial analogy to (4.6) we obtain that

$$\widehat{\chi} = L^{-1}\frac{dL}{dt} = \chi \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \widehat{\vartheta} = R^{-1}\frac{dR}{dt} = \vartheta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

where

$$\chi = \chi_{\mathrm{rl}} + \chi_{\mathrm{dr}} = \frac{d\alpha}{dt} + \cos\frac{r}{R}\frac{d\varphi}{dt} = \frac{d\alpha}{dt} + \cos\vartheta\frac{d\varphi}{dt},$$

but ϑ has no "drive" term, i.e.,

$$\vartheta = \frac{d\beta}{dt}.$$

Remark: do not confuse this ϑ with the angle ϑ.

It turns out that to avoid some embarrassing cross-terms it is convenient to introduce the "mixed" coordinates

$$x := \frac{1}{\sqrt{2}}(\lambda - \mu), \qquad y := \frac{1}{\sqrt{2}}(\lambda + \mu), \qquad \gamma := \alpha + \beta, \qquad \delta := \alpha - \beta.$$

The inverse rules read that

$$\lambda = \frac{1}{\sqrt{2}}(x+y), \qquad \mu = \frac{1}{\sqrt{2}}(y-x), \qquad \alpha = \frac{1}{2}(\gamma+\delta), \qquad \beta = \frac{1}{2}(\gamma-\delta).$$

The canonical momenta satisfy the contragradient rules

$$p_x = \frac{1}{\sqrt{2}}\left(p_\lambda - p_\mu\right), \qquad p_y = \frac{1}{\sqrt{2}}\left(p_\lambda + p_\mu\right),$$
$$p_\gamma = \frac{1}{2}\left(p_\alpha + p_\beta\right), \qquad p_\delta = \frac{1}{2}\left(p_\alpha - p_\beta\right),$$

and conversely,

$$p_\lambda = \frac{1}{\sqrt{2}}\left(p_x + p_y\right), \qquad p_\mu = \frac{1}{\sqrt{2}}\left(p_y - p_x\right),$$
$$p_\alpha = p_\gamma + p_\delta, \qquad p_\alpha = p_\gamma - p_\delta.$$

Remark: it is well known that the two-polar decomposition is not unique and has singular points [49]. Therefore, for $n = 2$ the $(\alpha, \beta, \lambda, \mu)$-parameterization is not a diffeomorphism between $\mathrm{GL}(2, \mathbb{R})$ and $\mathbb{T}^2 \times \mathbb{R}^2$, i.e., the Cartesian product of the two-dimensional torus and the real plane. Rather the representation manifold $\mathbb{T}^2 \times \mathbb{R}^2$ is plagued by a complicated system of identifications [49] and only the resulting quotient space may be interpreted as a proper representation of our configuration space. What concerns the quantities γ, δ the situation is even worse. They are "less proper" angular variables than α, β themselves are. The reason is that the integer-entries matrix transforming (α, β) into (γ, δ) has the determinant -2, thus, different from ± 1. If we normalize it to the ± 1 determinant, then its entries will be no longer integer. Therefore, this matrix is not a well-defined automorphism either of the torus $\mathbb{T}^2$ or the lattice $\mathbb{Z}^2$. The strange status of variables γ, δ is not misleading if it is not forgotten. And of course, it is quite proper to say simply that to avoid some cross-terms in the kinetic energy expression we replace the angular velocities χ, ϑ by their combinations, i.e.,

$$a := \chi + \vartheta, \qquad b := \chi - \vartheta.$$

Let us order our generalized coordinates q^i, $i = \overline{1, 6}$, as follows:

$$r, \varphi, \gamma, \delta, x, y.$$

Then the kinetic energy will be written as follows:

$$T = \frac{m}{2} G_{ij}(q) \frac{dq^i}{dt} \frac{dq^j}{dt},$$

where for the above ordering of variables the matrix $[G_{ij}]$ of the metric tensor G consists of three blocks subsequently placed along the diagonal (looking from the top to bottom):

(i) the 1×1 block M_1, i.e.,

$$M_1 = [1],$$

(ii) the 3×3 block M_2 given as follows:

$$M_2 = \begin{bmatrix} R^2 \sin^2 \frac{r}{R} + \frac{I}{m}\left(x^2 + y^2\right)\cos^2 \frac{r}{R} & \frac{I}{m}x^2 \cos \frac{r}{R} & \frac{I}{m}y^2 \cos \frac{r}{R} \\ \frac{I}{m}x^2 \cos \frac{r}{R} & \frac{I}{m}x^2 & 0 \\ \frac{I}{m}y^2 \cos \frac{r}{R} & 0 & \frac{I}{m}y^2 \end{bmatrix},$$

(iii) the 2×2 isotropic block M_3, i.e.,

$$M_3 = \frac{I}{m} I_2 = \begin{bmatrix} \frac{I}{m} & 0 \\ 0 & \frac{I}{m} \end{bmatrix},$$

where, obviously, I_2 denotes the 2×2 identity matrix.

It is seen that now, when deformative degrees of freedom are taken into account, the previous special case $I = mR^2$ does not lead to any essential simplification.

One can easily show that

$$\det [G_{ij}] = R^2 \left(\frac{I}{m}\right)^4 x^2 y^2 \sin^2 \frac{r}{R},$$

thus, the density of the Riemannian volume element equals

$$\sqrt{|G|} = R \left(\frac{I}{m}\right)^2 |xy| \sin \frac{r}{R}.$$

Explicitly, the block matrix $[G_{ij}]$ is given as follows:

$$[G_{ij}] = \begin{bmatrix} M_1 & & \\ & M_2 & \\ & & M_3 \end{bmatrix}.$$

The inverse contravariant tensor matrix $\left[G^{ij}\right]$ is obviously given by

$$\left[G^{ij}\right] = \begin{bmatrix} M_1^{-1} & & \\ & M_2^{-1} & \\ & & M_3^{-1} \end{bmatrix},$$

where the inverse blocks have the forms:

$(i)\ M_1^{-1} = [1]\,,$

$$(ii)\ M_2^{-1} = \begin{bmatrix} \frac{1}{R^2\sin^2\frac{r}{R}} & -\frac{\cos\frac{r}{R}}{R^2\sin^2\frac{r}{R}} & -\frac{\cos\frac{r}{R}}{R^2\sin^2\frac{r}{R}} \\ -\frac{\cos\frac{r}{R}}{R^2\sin^2\frac{r}{R}} & \frac{m}{I}\frac{1}{x^2}+\frac{1}{R^2}\mathrm{ctg}^2\frac{r}{R} & \frac{1}{R^2}\mathrm{ctg}^2\frac{r}{R} \\ -\frac{\cos\frac{r}{R}}{R^2\sin^2\frac{r}{R}} & \frac{1}{R^2}\mathrm{ctg}^2\frac{r}{R} & \frac{m}{I}\frac{1}{y^2}+\frac{1}{R^2}\mathrm{ctg}^2\frac{r}{R} \end{bmatrix},$$

$$(iii)\ M_3^{-1} = \tfrac{m}{I}I_2 = \begin{bmatrix} \frac{m}{I} & 0 \\ 0 & \frac{m}{I} \end{bmatrix}.$$

For potential systems with Lagrangians of the form

$$L = T - V(q),$$

the corresponding kinetic (geodetic) Hamiltonian equals

$$\mathcal{T} = \frac{1}{2m}G^{ij}(q)p_ip_j$$

and the full Hamiltonian is as follows:

$$H = \mathcal{T} + V(q).$$

According to our convention of ordering coordinates q^i, $i=\overline{1,6}$, i.e.,

$$r, \varphi, \gamma, \delta, x, y,$$

the corresponding conjugate momenta p_i, $i=\overline{1,6}$, are denoted and ordered as follows:

$$p_r, p_\varphi, p_\gamma, p_\delta, p_x, p_y.$$

In certain expressions it is convenient to use the original momenta p_α, p_β, p_λ, p_μ. First of all this concerns p_α, p_β because of their geometrical interpretation respectively as spin and vorticity [49].

For the above potential systems the reduced Hamilton-Jacobi equation has the form

$$\frac{1}{2m}G^{ij}(q)\frac{\partial S_0}{\partial q^i}\frac{\partial S_0}{\partial q^j} + V(q) = E.$$

We do not write it down explicitly because of the complicated and rather obscure form of the resulting expression. The above block form is much more readable and lucid.

Just as in the gyroscopic case we are dealing here with non-orthogonal coordinates (the 3×3 block M_2) and it is not clear for us whether in some hypothetic orthonormal coordinates (they exist, of course) the system is separable. It is perhaps a little surprising that our kinetic Hamiltonian $\mathcal{T}$ has the separable structure. Of course, for the system with deformative degrees of freedom as above, the geodetic model is not physical because it admits unlimited expansion and contraction. Therefore, some potential must be assumed and this is just the problem, i.e., we could not determine a wide class of potentials compatible with the separability in our non-orthogonal, but nevertheless natural, coordinates. Just as in the gyroscopic case we restrict ourselves to some special class of potentials, assuming in particular that all angles φ, α, β (equivalently φ, γ, δ) are cyclic variables. Therefore, the reduced action S_0 is sought as a function linear in angular variables and separable, i.e.,

$$\begin{aligned} S_0(q) &= S_r(r) + S_x(x) + S_y(y) + \ell\varphi + C_\gamma\gamma + C_\delta\delta \\ &= S_r(r) + S_x(x) + S_y(y) + \ell\varphi + C_\alpha\alpha + C_\beta\beta, \end{aligned} \tag{4.26}$$

where ℓ, C_γ, C_δ, C_α, C_β are constants. The relationship between (γ, δ) and (α, β) implies that

$$C_\alpha = C_\gamma + C_\delta, \qquad C_\beta = C_\gamma - C_\delta,$$

i.e.,

$$C_\gamma = \frac{1}{2}\left(C_\alpha + C_\beta\right), \qquad C_\delta = \frac{1}{2}\left(C_\alpha - C_\beta\right).$$

As p_α, p_β corresponding respectively to the spin and vorticity in the flat-space theory [49], we shall also denote the constants C_α, C_β as s, j, thus,

$$S_0(q) = S_r(r) + S_x(x) + S_y(y) + \ell\varphi + s\alpha + j\beta.$$

Let us now quote the expressions for the action variables, i.e.,

$$\begin{aligned} J_\alpha &= \oint p_\alpha d\alpha = C_\alpha \int_0^{2\pi} d\alpha = 2\pi C_\alpha = 2\pi s, \\ J_\beta &= \oint p_\beta d\beta = C_\beta \int_0^{2\pi} d\beta = 2\pi C_\beta = 2\pi j, \\ J_\varphi &= \oint p_\varphi d\varphi = \ell \int_0^{2\pi} d\varphi = 2\pi\ell. \end{aligned}$$

Let us assume that the potential energy separates explicitly with respect to acyclic variables, i.e.,

$$V(r,x,y) = V_r(r) + V_x(x) + V_y(y).$$

Then the resulting Hamilton-Jacobi equation also separates and denoting the corresponding separation constants by C_r, C_x, C_y we have that

$$C_r + C_x + C_y = E,$$

and then

$$\frac{1}{2m}\left(\frac{dS_r}{dr}\right)^2 + \frac{\left(J_\varphi - J_\alpha \cos\frac{r}{R}\right)^2}{8\pi^2 m R^2 \sin^2\frac{r}{R}} + V_r(r) = E - C_x - C_y, \tag{4.27}$$

$$\frac{1}{2I}\left(\frac{dS_x}{dx}\right)^2 + \frac{\left(J_\alpha + J_\beta\right)^2}{32\pi^2 I x^2} + V_x(x) = C_x, \tag{4.28}$$

$$\frac{1}{2I}\left(\frac{dS_y}{dy}\right)^2 + \frac{\left(J_\alpha - J_\beta\right)^2}{32\pi^2 I y^2} + V_y(y) = C_y. \tag{4.29}$$

Therefore, after expressing $C_\alpha = s$, $C_\beta = j$, $C_\varphi = \ell$ with the help of J_α, J_β, J_φ, we obtain for the remaining action variables the expressions

$$J_x = \oint p_x dx = \oint \sqrt{2I\left(C_x - V_x(x)\right) - \frac{\left(J_\alpha + J_\beta\right)^2}{16\pi^2 x^2}}dx, \tag{4.30}$$

$$J_y = \oint p_y dy = \oint \sqrt{2I\left(C_y - V_y(y)\right) - \frac{\left(J_\alpha - J_\beta\right)^2}{16\pi^2 y^2}}dy, \tag{4.31}$$

$$\begin{aligned} J_r &= \oint p_r dr \qquad (4.32) \\ &= \oint \sqrt{2m\left(E - C_x - C_y - V_r(r)\right) - \frac{\left(J_\varphi - J_\alpha \cos\frac{r}{R}\right)^2}{4\pi^2 R^2 \sin^2\frac{r}{R}}}dr. \end{aligned}$$

Due to the cyclic character of φ, α, β, the true dynamics is contained in (4.27), (4.28), (4.29) or on the level of action variables in (4.30), (4.31), (4.32). Obviously, the microdeformation dynamics is described by (4.28), (4.29) or equivalently by (4.31), (4.32). Unlike this, (4.27) and (4.32) describe the dynamical influence of the "North Pole" ($r = 0$) and "South Pole" ($r = \pi R$) on the translational motion. The dynamics of deformation invariants x, y is in many respects similar to the one in [37] (the combinations $\left(D_1 - D_2\right)/\sqrt{2}$ and $\left(D_1 + D_2\right)/\sqrt{2}$

in [37]). The r-geodetic model with $V_r = 0$ is obviously well formulated. But in d'Alembert models the (x, y)-geodetic case ($V_x = 0$, $V_y = 0$) would be quite not physical because of admitting unlimited expansion and contraction of the body. This is not the case in affine models where the "elastic vibrations" may be encoded in the very kinetic energy.

The procedure is now as follows. We assume some particular forms of the deformative potentials V_x, V_y basing both on some physical arguments (taken, e.g., from elasticity theory) and on the possibility of explicit analytical calculations (at least in terms of some familiar special functions) [37]. And later on, calculating (4.30), (4.31) and inverting the resulting formulas, we express C_x in terms of J_x, J_α, J_β and C_y in terms of J_y, J_α, J_β. These expressions show some kind of "degeneracy" because C_x depends on J_α, J_β through the integer combination $J_\alpha + J_\beta$, and similarly, C_y depends on them through $J_\alpha - J_\beta$. However, this does not prejudice the true degeneracy of the Hamiltonian. Substituting $C_x\left(J_x, J_\alpha + J_\beta\right)$ and $C_y\left(J_y, J_\alpha - J_\beta\right)$ to the main dynamical formula (4.32) and assuming again some functional form of V_r (e.g., just $V_r = 0$ in the translationally-free case), we in principle calculate J_r in terms of $E, J_\alpha, J_\beta, J_\varphi, J_x, J_y$. And again, solving this expression with respect to E, one finds in principle that

$$E = \mathcal{H}\left(J_r, J_\varphi, J_\alpha, J_\beta, J_x, J_y\right).$$

Without assuming the particular, both physical and analytically treatable, form of potentials one cannot a priori decide about problems concerning hyperintegrability (degeneracy) of the above integrable models.

Just as in the flat-space problems, there exist reasonably-looking models separable in other coordinates in the space of deformation invariants, moreover, separable simultaneously in several systems of coordinates in this space, thus probably degenerate (hyperintegrable) ones.

An interesting class of separable models is obtained when one uses the polar coordinates (ϱ, ε) in the space of deformation invariants:

$$x = \varrho \sin\varepsilon, \quad y = \varrho \cos\varepsilon. \tag{4.33}$$

Then the kinetic energy

$$T = \frac{m}{2} G_{ij}(q) \frac{dq^i}{dt} \frac{dq^j}{dt}$$

with coordinates ordered like

$$(q^1, q^2, q^3, q^4, q^5, q^6) = (r, \varphi, \gamma, \delta, \varrho, \varepsilon)$$

has the block matrix of the metric components

$$[G_{ij}] = \begin{bmatrix} K_1 & & \\ & K_2 & \\ & & K_3 \end{bmatrix},$$

where

$$K_1 = [1],$$

$$K_2 = \begin{bmatrix} R^2 \sin^2 \frac{r}{R} + \frac{I}{m}\varrho^2 \cos^2 \frac{r}{R} & \frac{I}{m}\varrho^2 \sin^2 \varepsilon \cos \frac{r}{R} & \frac{I}{m}\varrho^2 \cos^2 \varepsilon \cos \frac{r}{R} \\ \frac{I}{m}\varrho^2 \sin^2 \varepsilon \cos \frac{r}{R} & \frac{I}{m}\varrho^2 \sin^2 \varepsilon & 0 \\ \frac{I}{m}\varrho^2 \cos^2 \varepsilon \cos \frac{r}{R} & 0 & \frac{I}{m}\varrho^2 \cos^2 \varepsilon \end{bmatrix}, \tag{4.34}$$

$$K_3 = \begin{bmatrix} \frac{I}{m} & 0 \\ 0 & \frac{I}{m}\varrho^2 \end{bmatrix} = \frac{I}{m} \begin{bmatrix} 1 & 0 \\ 0 & \varrho^2 \end{bmatrix}.$$

The inverse metric tensor G^{ij} underlying the kinetic part of the Hamiltonian is given as follows:

$$\left[G^{ij}\right] = \begin{bmatrix} K_1^{-1} & & \\ & K_2^{-1} & \\ & & K_3^{-1} \end{bmatrix},$$

where, obviously,

$$K_1^{-1} = [1],$$

$$K_2^{-1} = \begin{bmatrix} \frac{1}{R^2 \sin^2 \frac{r}{R}} & -\frac{\cos \frac{r}{R}}{R^2 \sin^2 \frac{r}{R}} & -\frac{\cos \frac{r}{R}}{R^2 \sin^2 \frac{r}{R}} \\ -\frac{\cos \frac{r}{R}}{R^2 \sin^2 \frac{r}{R}} & \frac{m}{I}\frac{1}{\varrho^2 \sin^2 \varepsilon} + \frac{1}{R^2}\operatorname{ctg}^2 \frac{r}{R} & \frac{1}{R^2}\operatorname{ctg}^2 \frac{r}{R} \\ -\frac{\cos \frac{r}{R}}{R^2 \sin^2 \frac{r}{R}} & \frac{1}{R^2}\operatorname{ctg}^2 \frac{r}{R} & \frac{m}{I}\frac{1}{\varrho^2 \cos^2 \varepsilon} + \frac{1}{R^2}\operatorname{ctg}^2 \frac{r}{R} \end{bmatrix}, \tag{4.35}$$

$$K_3^{-1} = \begin{bmatrix} \frac{m}{I} & 0 \\ 0 & \frac{m}{I}\frac{1}{\varrho^2} \end{bmatrix} = \frac{m}{I} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\varrho^2} \end{bmatrix}.$$

Obviously, the purely kinetic (geodetic) Hamiltonian is given by

$$\mathcal{T} = \frac{1}{2m} G^{ij}(q) p_i p_j,$$

where the canonical momenta are ordered as follows:

$$(p_1, p_2, p_3, p_4, p_5, p_6) = (p_r, p_\varphi, p_\gamma, p_\delta, p_\varrho, p_\varepsilon).$$

Strictly speaking, our "polar coordinates" ϱ, ε are rotated by $\pi/2$ with respect to the usual convention,but for some reasons it is convenient for us.

Tensor densities built of the metric G_{ij} are in these coordinates given by the expressions

$$\begin{aligned} G &= \det\left[G_{ij}\right] = R^2\left(\frac{I}{m}\right)^4 \varrho^4 \sin^2\varepsilon\cos^2\varepsilon\sin^2\frac{r}{R}, \\ \sqrt{G} &= R\left(\frac{I}{m}\right)^2 \varrho^2\left|\sin\varepsilon\cos\varepsilon\right|\sin\frac{r}{R}. \end{aligned}$$

As previously, the system is separable (thus, completely integrable) for potentials independent of φ, α, β. Then the reduced action S_0 is given by

$$S_0(q) = S_r(r) + \ell\varphi + C_\alpha\alpha + C_\beta\beta + S_\varrho(\varrho) + S_\varepsilon(\varepsilon),$$

where as previously ℓ, C_α, C_β are integration constants. Using again the terms of spin and vorticity s, j we can write that

$$S_0(q) = S_r(r) + \ell\varphi + s\alpha + j\beta + S_\varrho(\varrho) + S_\varepsilon(\varepsilon).$$

It is easily seen that such problems with cyclic variables φ, α, β are separable for deformation potentials of the form

$$V_\varrho(\varrho) + \frac{V_\varepsilon(\varepsilon)}{\varrho^2}, \tag{4.36}$$

i.e., for the total potentials we have that

$$V(r,\varrho,\varepsilon) = V_r(r) + V_\varrho(\varrho) + \frac{V_\varepsilon(\varepsilon)}{\varrho^2}. \tag{4.37}$$

Just as previously the action variables J_α, J_β, J_φ are given as follows:

$$\begin{aligned} J_\alpha &= \oint p_\alpha d\alpha = C_\alpha\int_0^{2\pi} d\alpha = 2\pi C_\alpha = 2\pi s, \\ J_\beta &= \oint p_\beta d\beta = C_\beta\int_0^{2\pi} d\beta = 2\pi C_\beta = 2\pi j, \\ J_\varphi &= \oint p_\varphi d\varphi = \ell\int_0^{2\pi} d\varphi = 2\pi\ell. \end{aligned}$$

Then S_ϱ, S_ε, S_r satisfy the ordinary differential equations

$$\begin{aligned}
\varrho^2\left(\frac{dS_\varrho}{d\varrho}\right)^2 + 2I\varrho^2(V_\varrho(\varrho)-C) &= -2IA,\\
\left(\frac{dS_\varepsilon}{d\varepsilon}\right)^2 + \frac{J_\alpha^2+2J_\alpha J_\beta\cos 2\varepsilon + J_\beta^2}{4\pi^2\sin^2 2\varepsilon} + 2IV_\varepsilon(\varepsilon) &= 2IA,\\
\frac{1}{2m}\left(\frac{dS_r}{dr}\right)^2 + \frac{(J_\varphi - J_\alpha\cos\frac{r}{R})^2}{8\pi^2 mR^2\sin^2\frac{r}{R}} &= E-C-V_r(r),
\end{aligned}$$

where C and A are the separation constants. More precisely, $2IA$ is the constant value of the Hamilton-Jacobi term depending only on ε, and $(E-C)$ is the constant value of the r-term. Therefore, the remaining action quantities are given by:

$$J_\varepsilon = \oint\sqrt{2I(A-V_\varepsilon(\varepsilon)) - \frac{J_\alpha^2+2J_\alpha J_\beta\cos 2\varepsilon + J_\beta^2}{4\pi^2\sin^2 2\varepsilon}}\,d\varepsilon, \tag{4.38}$$

$$J_\varrho = \oint\sqrt{2I(C-V_\varrho(\varrho)) - \frac{2IA}{\varrho^2}}\,d\varrho, \tag{4.39}$$

$$J_r = \oint\sqrt{2m(E-C-V_r(r)) - \frac{\left(J_\varphi - J_\alpha\cos\frac{r}{R}\right)^2}{4\pi^2R^2\sin^2\frac{r}{R}}}\,dr. \tag{4.40}$$

And again the calculation procedure is as follows. Some explicit forms of controlling potential functions V_ε, V_ϱ, V_r must be specified on the basis of microphysical, phenomenological, or simply geometrical arguments. Later on one "calculates" the integral (4.38) and expresses J_ε as a function of A, J_α, J_β, i.e.,

$$J_\varepsilon = J_\varepsilon\left(A, J_\alpha, J_\beta\right).$$

Solving this expression with respect to A, one obtains the formula for A as a function of three action variables:

$$A = A\left(J_\alpha, J_\beta, J_\varepsilon\right).$$

Substituting this to (4.39) and "calculating" the integral, one obtains J_ϱ as a function of C, J_α, J_β, J_ε, i.e.,

$$J_\varrho = J_\varrho\left(C, J_\alpha, J_\beta, J_\varepsilon\right).$$

Solving this with respect to C, one can in principle express C as function of four action variables:

$$C = C\left(J_\alpha, J_\beta, J_\varrho, J_\varepsilon\right).$$

And finally, this expression is substituted to (4.40) and then we obtain the formula for J_r as a function of E, J_α, J_β, J_ϱ, J_ε, J_φ, i.e.,

$$J_r = J_r(E, J_\alpha, J_\beta, J_\varrho, J_\varepsilon, J_\varphi).$$

Solving this with respect to E, one obtains the concluding formula expressing E through the (six) action variables:

$$E = \mathcal{H}(J_r, J_\varphi, J_\alpha, J_\beta, J_\varrho, J_\varepsilon).$$

Unfortunately, unlike in the mechanics of material point moving in a central field, even the partial degeneracy cannot be concluded directly from the formulas (4.38), (4.39), (4.40) without performing the integration process based on some fixed forms of the potential controlling functions V_ε, V_ϱ, V_r. And just (and even more so) like in the mechanics of infinitesimal gyroscope, interaction with internal degrees of freedom removes the total degeneracy of the material-point Bertrand models (4.3), (4.4).

The potentials of the form (4.36), (4.37) are very convenient from the point of view of nonlinear macroscopic elasticity. Being compatible with the very nature of deformative degrees of freedom they are also interesting in the theory of infinitesimal objects. The following example is very instructive, moreover in macroscopic nonlinear elasticity in two dimensions it is almost canonical [49]:

$$V = \frac{2\varkappa}{\varrho^2 \cos 2\varepsilon} + \frac{\varkappa}{2}\varrho^2 = \varkappa\left(\frac{1}{\lambda\mu} + \frac{\lambda^2+\mu^2}{2}\right), \qquad \varkappa > 0. \tag{4.41}$$

The first term prevents any kind of collapse of the two-dimensional body: to the point or to the straight line. The second term of the "harmonic oscillator" type prevents the unlimited expansion. The natural state $\lambda = \mu = 1$ (no deformation) minimizes the potential energy V; it is a stable equilibrium. Extension in one direction is accompanied by contraction in the orthogonal one. One can invent plenty of similar potentials just using the polar coordinates ϱ, ε in the space of deformation invariants. The above one is particularly simple and suggestive; it is also well suited to the analytical procedure.

4.2 Pseudospherical case

Let us now consider the same problems on the pseudosphere, i.e., on the two-dimensional Lobatshevski space. We assume it to have the pseudoradius R, therefore, the constant negative curvature $\mathcal{R} = -2/R^2$. The most natural realization of this space is the "upper" shell of the hyperboloid in $\mathbb{R}^3$, i.e.,

$$-x^2 - y^2 + z^2 = R^2, \qquad z > 0.$$

We denote it as $H^{2,2,+}(0,R)$. Its metric tensor is obtained as the restriction (injection-pull-back) of the Minkowski metric:

$$dx^2 + dy^2 - dz^2.$$

Again we introduce the "polar" coordinates (r, φ), where r is the geodetic distance measured from the "North Pole" $x = 0$, $y = 0$, $z = R$, and φ is the "geographic longitude", i.e., the polar angle in the sense of the plane $z = 0$.
Remark: the "South Pole" $x = 0$, $y = 0$, $z = -R$ is not interesting for us as it is placed on the "lower" shell of the hyperboloid, i.e., on the other connected component. $H^{2,2,+}(0,R)$ is not compact and the radial variable r has the infinite range $[0, \infty]$. We shall also use the pseudoangle $\vartheta = r/R$. The parametric description of $H^{2,2,+}(0,R) \subset \mathbb{R}^3$ is given by

$$x = R\mathrm{sh}\vartheta \cos\varphi, \qquad y = R\mathrm{sh}\vartheta \sin\varphi, \qquad z = R\mathrm{ch}\vartheta.$$

Expressed in terms of coordinates (r, φ), the metric element of $H^{2,2,+}(0,R)$, in analogy to (4.1), has the form

$$ds^2 = dr^2 + R^2 \mathrm{sh}^2 \frac{r}{R} d\varphi^2.$$

And in general, practically all pseudospherical formulas may be obtained from the spherical ones by substituting hyperbolic functions instead of the corresponding trigonometric ones. In certain expressions one must be however careful with signs. So, the translational kinetic energy, in analogy to (4.2), has the form

$$T_{\mathrm{tr}} = \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + \mathrm{sh}^2\frac{r}{R}\left(\frac{d\varphi}{dt}\right)^2\right).$$

And again there are two Bertrand-type potentials [35, 40], i.e.,

(i) the "harmonic oscillator"-type potential:

$$V(r) = \frac{\varkappa}{2} R^2 \mathrm{th}^2 \frac{r}{R}, \qquad \varkappa > 0, \tag{4.42}$$

(ii) the "attractive Kepler-Coulomb"-type one:

$$V(r) = -\frac{\alpha}{R}\mathrm{cth}\frac{r}{R}, \qquad \alpha > 0. \tag{4.43}$$

With these and only these potentials all bounded orbits are closed. And now the term "bounded" is essential because the "physical space" is now not compact.

And indeed, there exist unbounded motions corresponding to energy values exceeding some thresholds. It is interesting that unlike in the spherical world, in Lobatshevski space the isotropic degenerate oscillator has an open subset of unbounded trajectories because the potential (4.42) has a finite upper bound, i.e.,

$$\mathrm{Sup}\ V = \frac{\varkappa}{2}R^2.$$

For energy values above this threshold all trajectories are unbounded, the motion is infinite. Below this threshold all trajectories are not only bounded but also periodic.

The existence of threshold in (4.43) is not surprising, it is like in the usual Kepler in $\mathbb{R}^2$. But the threshold for the isotropic degenerate oscillator is a very interesting feature of the Lobatshevski "world".

Let us now consider an infinitesimal gyroscope. The success of (4.5) suggests us to use the orthonormal basis

$$E_r = \frac{\partial}{\partial r} = \mathcal{E}_r, \qquad E_\varphi = \frac{1}{R\mathrm{sh}\frac{r}{R}}\frac{\partial}{\partial \varphi} = \frac{1}{R\mathrm{sh}\frac{r}{R}}\mathcal{E}_\varphi,$$

or in terms of components:

$$E_r = [1,0]^T, \qquad E_\varphi = \frac{1}{R\mathrm{sh}\frac{r}{R}}[0,1]^T.$$

It is easy to see that this aholonomic basis is really orthonormal because

$$g_{r\varphi} = g_{\varphi r} = 0, \qquad g_{rr} = 1, \qquad g_{\varphi\varphi} = R^2\mathrm{sh}^2\frac{r}{R}.$$

We formally repeat all considerations of the spherical case, i.e., all formulas for Christoffel symbols are analogous, the trigonometric functions are replaced by the hyperbolic ones, etc. Preserving the same notation we obtain that

$$\omega_{\mathrm{rl}} = \frac{d\psi}{dt}, \qquad \omega = \omega_{\mathrm{rl}} + \omega_{\mathrm{dr}} = \frac{d\psi}{dt} + \mathrm{ch}\frac{r}{R}\frac{d\varphi}{dt} = \frac{d\psi}{dt} + \mathrm{ch}\vartheta\frac{d\varphi}{dt}.$$

In analogy to (4.7), (4.8) we obtain that

$$\begin{aligned} T &= T_{\mathrm{tr}} + T_{\mathrm{int}} \\ &= \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + R^2\mathrm{sh}^2\frac{r}{R}\left(\frac{d\varphi}{dt}\right)^2\right) + \frac{I}{2}\left(\frac{d\psi}{dt} + \mathrm{ch}\frac{r}{R}\frac{d\varphi}{dt}\right)^2, \end{aligned} \tag{4.44}$$

i.e.,

$$\begin{aligned} T &= T_{\mathrm{tr}} + T_{\mathrm{int}} \\ &= \frac{mR^2}{2}\left(\left(\frac{d\vartheta}{dt}\right)^2 + \mathrm{sh}^2\vartheta\left(\frac{d\varphi}{dt}\right)^2\right) + \frac{I}{2}\left(\frac{d\psi}{dt} + \mathrm{ch}\vartheta\frac{d\varphi}{dt}\right)^2. \end{aligned} \tag{4.45}$$

When the generalized coordinates (q^1, q^2, q^3) are ordered as previously, i.e, (r, φ, ψ), then the corresponding metric on the configuration space is given as follows:

$$[G_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2\left(\mathrm{sh}^2\frac{r}{R} + \frac{I}{mR^2}\mathrm{ch}^2\frac{r}{R}\right) & \frac{I}{m}\mathrm{ch}\frac{r}{R} \\ 0 & \frac{I}{m}\mathrm{ch}\frac{r}{R} & \frac{I}{m} \end{bmatrix}. \tag{4.46}$$

The kinetic energy is then given by the expression

$$T = \frac{m}{2}G_{ij}(q)\frac{dq^i}{dt}\frac{dq^j}{dt}.$$

The contravariant metric $\left[G^{ij}\right]$ has the form

$$\left[G^{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{R^2\mathrm{sh}^2\frac{r}{R}} & -\frac{\mathrm{ch}\frac{r}{R}}{R^2\mathrm{sh}^2\frac{r}{R}} \\ 0 & -\frac{\mathrm{ch}\frac{r}{R}}{R^2\mathrm{sh}^2\frac{r}{R}} & \frac{m}{I} + \frac{1}{R^2}\mathrm{cth}^2\frac{r}{R} \end{bmatrix}.$$

For potential systems the corresponding geodetic Hamiltonian is written as follows:

$$\mathcal{T} = \frac{1}{2m}G^{ij}(q)p_i p_j,$$

where, obviously, the generalized momenta (p_1, p_2, p_3) are ordered as follows: (p_r, p_φ, p_ψ).

The weight-two density built of G has the form

$$G = \det\left[G_{ij}\right] = R^2\frac{I}{m}\mathrm{sh}^2\frac{r}{R}, \tag{4.47}$$

and the weight-one density defining the volume element is given by

$$\sqrt{G} = R\sqrt{\frac{I}{m}}\mathrm{sh}\frac{r}{R}. \tag{4.48}$$

It is seen that the spherically very special case $I = mR^2$ here, in the pseudo-spherical case also leads to some simplification of $[G_{ij}]$, but not so striking one as previously. This fact has deep geometric reasons which will be explained in the sequel.

Namely, in the spherical space an essential point is the natural identification between the quotient manifold $\mathrm{SO}(3,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ and the spheres $S^2(0,R)$, $S^2(0,1)$. And this has to do with the formal identification between two-dimensional rigid body moving over the spherical surface and the three-dimensional symmetrical top without translational degrees of freedom. The special case $I = mR^2$ corresponds to the spherical top. In general the kinetic energy is then invariant under $\mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(2,\mathbb{R})$. In the three-dimensional top analogy $\mathrm{SO}(3,\mathbb{R})$ is acting as left regular translations and $\mathrm{SO}(2,\mathbb{R})$ as right regular translations corresponding to the group of rotations around the body-fixed z-axis. If $I = mR^2$ we have the full invariance under $\mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(3,\mathbb{R})$.

In the hyperbolic pseudospherical geometry the problem is isomorphic with the three-dimensional Lorentzian (Minkowskian) top on $\mathbb{R}^3$. The rotation group $\mathrm{SO}(3,\mathbb{R})$ is replaced by the three-dimensional Lorentz group $\mathrm{SO}(1,2)$. And still an important role is played by $\mathrm{SO}(2,\mathbb{R})$ interpreted again as the group of usual rotations in Euclidean space of (x,y)-variables (thus, not affecting the z-direction). The above kinetic energy (4.44), (4.45) is invariant under $\mathrm{SO}(1,2) \times \mathrm{SO}(2,\mathbb{R})$. But it is never invariant under $\mathrm{SO}(1,2) \times \mathrm{SO}(1,2)$, i.e., under left and right Lorentz regular translations in the $\mathrm{SO}(1,2)$-sense. The spherical special case $I = mR^2$ does not help here. Indeed, the underlying metric G (and the kinetic energy itself) is positively definite. But the doubly-invariant ($\mathrm{SO}(1,2) \times \mathrm{SO}(1,2)$-invariant) metric on $\mathrm{SO}(1,2)$, i.e., its Killing metric is not positively definite. Instead it has the normal-hyperbolic signature $(++-)$. The reason is that it is semi-simple (even simple) non-compact group. This brings about the question about non-positive kinetic energies (metric tensors) on our configuration space. As the negative contribution to the Killing metric tensor on $\mathrm{SO}(1,2)$ comes from its compact subgroup $\mathrm{SO}(2,\mathbb{R})$ of (x,y)-rotations, i.e., from the gyroscopic degree of freedom in the language of $H^{2,2,+}(0,R)$, there is a natural suggestion to invert the sign of the gyroscopic contribution to (4.7), (4.8), i.e., to make it negative. One is naturally reluctant to indefinite kinetic energies but there are examples when they are just convenient and very useful as tools for describing some kinds of physical interactions [42, 44, 45, 46, 47, 48, 49, 50, 51], just encoding them even without any use of potentials.

So, we can try to use, or at least mathematically analyze, the "Lorentz-type kinetic energies" T_L of the form

$$
\begin{aligned}
T_L &= \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + R^2 \mathrm{sh}^2\frac{r}{R}\left(\frac{d\varphi}{dt}\right)^2\right) - \frac{I}{2}\left(\frac{d\psi}{dt} + \mathrm{ch}\frac{r}{R}\frac{d\varphi}{dt}\right)^2 \\
&= \frac{mR^2}{2}\left(\left(\frac{d\vartheta}{dt}\right)^2 + \mathrm{sh}^2\vartheta\left(\frac{d\varphi}{dt}\right)^2\right) - \frac{I}{2}\left(\frac{d\psi}{dt} + \mathrm{ch}\vartheta\frac{d\varphi}{dt}\right)^2. \qquad (4.49)
\end{aligned}
$$

Thus, it is so as if the extra rotation diminished effectively the kinetic energy of translational motion. If we write as usual that

$$T_L = \frac{m}{2} {}_L G_{ij}(q) \frac{dq^i}{dt}\frac{dq^j}{dt},$$

then, with the same as previously convention concerning the ordering of coordinates (r, φ, ψ), we have that

$$[{}_L G_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2\left(\mathrm{sh}^2\frac{r}{R} - \frac{I}{mR^2}\mathrm{ch}^2\frac{r}{R}\right) & -\frac{I}{m}\mathrm{ch}\frac{r}{R} \\ 0 & -\frac{I}{m}\mathrm{ch}\frac{r}{R} & -\frac{I}{m} \end{bmatrix}$$

(compare this with (4.46)). And now, obviously, the remarkable simplification occurs in the very special case $I = mR^2$ just as in the spherical symmetry. This has to do "as usual" with the enlarging of the symmetry group from $\mathrm{SO}(1,2) \times \mathrm{SO}(2,\mathbb{R})$ to $\mathrm{SO}(1,2) \times \mathrm{SO}(1,2)$ (two additional parameters of symmetry). And namely, ${}_L G$ becomes then ${}_L \breve{G}$, i.e.,

$$\left[{}_L \breve{G}_{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -R^2 & -R^2\mathrm{ch}\frac{r}{R} \\ 0 & -R^2\mathrm{ch}\frac{r}{R} & -R^2 \end{bmatrix}$$

(compare this with (4.16) and notice the characteristic sign differences).

The weight-two scalar density built of ${}_L G$ has the form

$$\det[{}_L G_{ij}] = -R^2 \frac{I}{m}\mathrm{sh}^2\frac{r}{R},$$

so it differs in sign from (4.47). This difference obviously does not influence the density of Riemannian volume element, i.e.,

$$\sqrt{|\det[{}_L G_{ij}]|} = R\sqrt{\frac{I}{m}}\mathrm{sh}\frac{r}{R},$$

exactly as in (4.48). This fact is of some interest.

As usual, the geodetic Hamiltonian is given by the expression

$$\mathcal{T}_L = \frac{1}{2m} {}_L G^{ij}(q) p_i p_j,$$

where ${}_LG^{ij}$ are components of the contravariant inverse of ${}_LG_{ij}$. They are given as follows:

$$\left[{}_LG^{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ & & \\ 0 & \frac{1}{R^2\mathrm{sh}^2\frac{r}{R}} & -\frac{\mathrm{ch}\frac{r}{R}}{R^2\mathrm{sh}^2\frac{r}{R}} \\ & & \\ 0 & -\frac{\mathrm{ch}\frac{r}{R}}{R^2\mathrm{sh}^2\frac{r}{R}} & -\frac{m}{I} + \frac{1}{R^2}\mathrm{cth}^2\frac{r}{R} \end{bmatrix}.$$

Obviously, if we use the above isomorphism between the two-dimensional top sliding over the Lobatshevski plane with the three-dimensional Lorentz top without translational motion in $\mathbb{R}^3$, then it is clear that ${}_LG$ is, up to normalization, identical with the Killing metric tensor of $\mathrm{SO}(1,2)$. Let us quote some formulas and concepts analogous to three-dimensional angular velocities, i.e., to (4.10), (4.11). And then the kinetic energy will be expressed like in (4.15).

First of all we parameterize $\mathrm{SO}(1,2)$ with the help of what we call the "pseudo-Euler angles". So, let us write that

$$\mathrm{SO}(1,2) \ni L(\varphi,\vartheta,\psi) = U_z(\varphi)L_x(\vartheta)U_z(\psi),$$

where the meaning of U_z is like in (4.9) and L_x denotes some Lorentz transformation in $\mathbb{R}^3$, namely, the "boost" along the x-axis, i.e.,

$$L_x(\vartheta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{ch}\vartheta & \mathrm{sh}\vartheta \\ 0 & \mathrm{sh}\vartheta & \mathrm{ch}\vartheta \end{bmatrix}.$$

During the motion all these quantities are functions of time and we can calculate the corresponding Lie-algebraic element

$$\widehat{\lambda} = L^{-1}\frac{dL}{dt},$$

i.e., the co-moving pseudo-angular velocity. After some calculations we obtain formulas analogous to (4.11), (4.12), (4.13), (4.14), and namely,

$$\widehat{\lambda} = \widehat{\lambda}_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \widehat{\lambda}_2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \widehat{\lambda}_3 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} \widehat{\lambda}_1 &= \mathrm{sh}\vartheta\sin\psi\frac{d\varphi}{dt} + \cos\psi\frac{d\vartheta}{dt}, \\ \widehat{\lambda}_2 &= -\mathrm{sh}\vartheta\cos\psi\frac{d\varphi}{dt} + \sin\psi\frac{d\vartheta}{dt}, \\ \widehat{\lambda}_3 &= \mathrm{ch}\vartheta\frac{d\varphi}{dt} + \frac{d\psi}{dt}. \end{aligned}$$

Both the similarities and differences in comparison with the corresponding spherical formulas are easily seen.

And now we can write two formulas analogous to (4.15), i.e.,

$$T = \frac{K}{2}\left(\widehat{\lambda}_1\right)^2 + \frac{K}{2}\left(\widehat{\lambda}_2\right)^2 + \frac{I}{2}\left(\widehat{\lambda}_3\right)^2, \tag{4.50}$$

$$T = \frac{K}{2}\left(\widehat{\lambda}_1\right)^2 + \frac{K}{2}\left(\widehat{\lambda}_2\right)^2 - \frac{I}{2}\left(\widehat{\lambda}_3\right)^2, \tag{4.51}$$

where $K > 0$ and $I > 0$. This is the symmetric SO$(1,2)$-top in $\mathbb{R}^3$. The indefinite expression (4.51) is structurally suited to the normal-hyperbolic signature of SO$(1,2)$. When $K = I$, then it becomes the spherical Lorentz top in $\mathbb{R}^3$ in the indefinite version based on the Killing metric.

It is easily seen that both expressions (4.50) and (4.51) are invariant under SO$(1,2) \times$ SO$(2,\mathbb{R})$, where SO$(1,2)$ and SO$(2,\mathbb{R})$ acts on SO$(1,2)$ through respectively the left and right regular translations. The form (4.51) with $K = I$ is invariant under all regular translations (both left and right), i.e., under SO$(1,2) \times$ SO$(1,2)$. And specifying $K = mR^2$ in (4.50) and (4.51), we obtain respectively (4.44) and (4.49).

Let us now turn to the Hamilton-Jacobi equation and action-angle variables. Just as previously, φ, ψ are cyclic variables in the kinetic energy term and we assume that the same is true for the total Hamiltonian, i.e., that the potential energy depends only on the variable r. Because of the non-diagonal structure of $[G_{ij}]$ and $[{}_LG_{ij}]$ in the natural variables (r, φ, ψ), it is rather difficult to decide a priori what would be (if any!) the form of the general separable potential. So, just as in (4.19) the reduced action S_0 is sought (with the same meaning of all symbols) in the form

$$S_0(r, \varphi, \psi; E, \ell, s) = S_r(r, E) + \ell\varphi + s\psi.$$

Just as in (4.20) the genuine dynamics reduces to the radial function which satisfies the equation

$$\left(\frac{dS_r}{dr}\right)^2 = 2m\left(E - V(r)\right) - \frac{\left(\ell - s\,\mathrm{ch}\frac{r}{R}\right)^2}{R^2\mathrm{sh}^2\frac{r}{R}} \pm \frac{m}{I}s^2. \tag{4.52}$$

The $\pm$ signs in (4.52) refer respectively to the models (4.44), (4.49). Just as in the spherical case p_φ, p_ψ are constants of motion and their constant values on fixed trajectories equal

$$p_\varphi = \frac{\partial S_0}{\partial \varphi} = \ell, \qquad p_\psi = \frac{\partial S_0}{\partial \psi} = s.$$

And, of course, their action variables are given as follows:

$$J_\varphi = \oint p_\varphi d\varphi = \int_0^{2\pi} \ell d\varphi = 2\pi\ell,$$
$$J_\psi = \oint p_\psi d\psi = \int_0^{2\pi} s d\psi = 2\pi s.$$

Substituting this to the radial action expression and to (4.52) we obtain for

$$J_r = \oint p_r dr$$

the following formula:

$$J_r = \oint \sqrt{2m(E - V(r)) - \frac{\left(J_\varphi - J_\psi \mathrm{ch}\frac{r}{R}\right)^2}{4\pi^2 R^2 \mathrm{sh}^2 \frac{r}{R}} \pm \frac{m}{I}\frac{J_\psi^2}{4\pi^2}} dr \qquad (4.53)$$

(with the above-mentioned meaning of the $\pm$ signs).

From now on the procedure is just as previously. After substituting here some particular form of V we perform the above contour integration and (at least in principle) determine the dependence of J_r on E, J_φ, J_ψ, i.e.,

$$J_r = J_r(E, J_\varphi, J_\psi).$$

Solving this equation with respect to E, one obtains the final expression for Hamiltonian:

$$E = \mathcal{H}(J_r, J_\varphi, J_\psi).$$

Just as in the spherical case no degeneracy structure may be directly deduced from (4.53) even in the special case of the minus sign and $I = mR^2$.

An important point is that, because of the uncompactness of the Lobatshevski space, the geodetic motion ($V = 0$) is unbounded, and then obviously the action-angle formalism becomes meaningless. Nevertheless, even for unbounded motion the isotropic models with V depending only on r may be effectively studied with the use of explicitly known constants of motion and reduced to first-order ordinary differential equations.

Let us now consider a deformable top moving in Lobatshevski space. All symbols concerning internal degrees of freedom are just those used in spherical geometry. The metric tensor G underlying the kinetic energy expression, i.e.,

$$T = \frac{m}{2} G_{ij}(q) \frac{dq^i}{dt}\frac{dq^j}{dt},$$

has the form analogous to the spherical case with the trigonometric functions simply replaced by the hyperbolic ones without any change of sign. Thus, the matrix $[G_{ij}]$ consists of three blocks M_1, M_2, M_3, where

$$M_1 = [1],$$

$$M_2 = \begin{bmatrix} R^2 \mathrm{sh}^2 \frac{r}{R} + \frac{I}{m}\left(x^2 + y^2\right)\mathrm{ch}^2 \frac{r}{R} & \frac{I}{m} x^2 \mathrm{ch} \frac{r}{R} & \frac{I}{m} y^2 \mathrm{ch} \frac{r}{R} \\ \frac{I}{m} x^2 \mathrm{ch} \frac{r}{R} & \frac{I}{m} x^2 & 0 \\ \frac{I}{m} y^2 \mathrm{ch} \frac{r}{R} & 0 & \frac{I}{m} y^2 \end{bmatrix},$$

$$M_3 = \frac{I}{m} I_2 = \begin{bmatrix} \frac{I}{m} & 0 \\ 0 & \frac{I}{m} \end{bmatrix}.$$

When deformations are admitted, then there is no particular geometric motivation for discussing the "Lorentzian" model with the indefinite kinetic energy. This is nevertheless possible and formally simply consists in replacing I by $-I$ in all formulas; in other words negative "inertial moments" are admitted.

For the inverse contravariant metric $\left[G^{ij}\right]$ underlying the geodetic Hamiltonian, i.e.,

$$\mathcal{T} = \frac{1}{2m} G^{ij}(q) p_i p_j,$$

we have the block structure also quite analogous to the spherical formulas:

$$M_1^{-1} = [1],$$

$$M_2^{-1} = \begin{bmatrix} \frac{1}{R^2 \mathrm{sh}^2 \frac{r}{R}} & -\frac{\mathrm{ch} \frac{r}{R}}{R^2 \mathrm{sh}^2 \frac{r}{R}} & -\frac{\mathrm{ch} \frac{r}{R}}{R^2 \mathrm{sh}^2 \frac{r}{R}} \\ -\frac{\mathrm{ch} \frac{r}{R}}{R^2 \mathrm{sh}^2 \frac{r}{R}} & \frac{m}{I}\frac{1}{x^2} + \frac{1}{R^2} \mathrm{cth}^2 \frac{r}{R} & \frac{1}{R^2} \mathrm{cth}^2 \frac{r}{R} \\ -\frac{\mathrm{ch} \frac{r}{R}}{R^2 \mathrm{sh}^2 \frac{r}{R}} & \frac{1}{R^2} \mathrm{cth}^2 \frac{r}{R} & \frac{m}{I}\frac{1}{y^2} + \frac{1}{R^2} \mathrm{cth}^2 \frac{r}{R} \end{bmatrix},$$

$$M_3^{-1} = \frac{m}{I} I_2 = \begin{bmatrix} \frac{m}{I} & 0 \\ 0 & \frac{m}{I} \end{bmatrix}.$$

The corresponding scalar G-densities are given as follows:

$$\begin{aligned} G = \det\left[G_{ij}\right] &= R^2 \left(\frac{I}{m}\right)^4 x^2 y^2 \mathrm{sh}^2 \frac{r}{R}, \\ \sqrt{|G|} &= R \left(\frac{I}{m}\right)^2 |xy| \,\mathrm{sh} \frac{r}{R}. \end{aligned}$$

Again we assume that the potential energy does not depend on the angles φ, α, β, i.e., they are cyclic variables for the total Hamiltonian. So, just as previously, the complete integral of

$$\frac{1}{2m}G^{ij}(q)\frac{\partial S_0}{\partial q^i}\frac{\partial S_0}{\partial q^j} + V(q) = E$$

will be sought as a linear function of all angles, i.e.,

$$\begin{aligned} S_0(q) &= S_r(r) + S_x(x) + S_y(y) + \ell\varphi + C_\gamma\gamma + C_\delta\delta \\ &= S_r(r) + S_x(x) + S_y(y) + \ell\varphi + C_\alpha\alpha + C_\beta\beta, \end{aligned}$$

with the same as previously relationships between constants. Coming back to the more familiar spin and vorticity symbols s, j, we have that $C_\alpha = s$ and $C_\beta = j$. The action variables corresponding to J_α, J_β, J_φ are respectively given by the same formulas like in the spherical geometry:

$$\begin{aligned} J_\alpha &= \oint p_\alpha d\alpha = C_\alpha \int_0^{2\pi} d\alpha = 2\pi C_\alpha = 2\pi s, \\ J_\beta &= \oint p_\beta d\beta = C_\beta \int_0^{2\pi} d\beta = 2\pi C_\beta = 2\pi j, \\ J_\varphi &= \oint p_\varphi d\varphi = \ell \int_0^{2\pi} d\varphi = 2\pi\ell. \end{aligned}$$

Similarly, when the (x, y)-deformation invariants are used, the most natural separable potentials have the explicitly separated form:

$$V(r, x, y) = V_r(r) + V_x(x) + V_y(y).$$

Then, just as in the spherical case, denoting the separation constants by C_r, C_x, and C_y, we have that

$$C_r + C_x + C_y = E$$

and the expressions for J_r, J_x, J_y just analogous to (4.30), (4.31), (4.32), namely,

$$\begin{aligned} J_x &= \oint p_x dx = \oint \sqrt{2I(C_x - V_x(x)) - \frac{(J_\alpha + J_\beta)^2}{16\pi^2 x^2}}dx, \\ J_y &= \oint p_y dy = \oint \sqrt{2I(C_y - V_y(y)) - \frac{(J_\alpha - J_\beta)^2}{16\pi^2 y^2}}dy, \\ J_r &= \oint p_r dr = \oint \sqrt{2m(E - C_x - C_y - V_r(r)) - \frac{(J_\varphi - J_\alpha ch\frac{r}{R})^2}{4\pi^2 R^2 sh^2\frac{r}{R}}}dr. \end{aligned}$$

The procedure of eliminating constants C_x, C_y and expressing E through the action variables, i.e.,

$$E = \mathcal{H}(J_r, J_\varphi, J_\alpha, J_\beta, J_x, J_y),$$

proceeds exactly as in the spherical case.

Exactly as in the theory of deformable gyroscope in the spherical space it is convenient and practically useful to parameterize deformation invariants with the use of polar variables ϱ, ε (see (4.33)). The only formal difference is that the trigonometric functions of r/R (but not those of ε!) are replaced by the hyperbolic ones without the change of sign, thus, (4.34) and (4.35) take on respectively the forms

$$K_2 = \begin{bmatrix} R^2\mathrm{sh}^2\frac{r}{R} + \frac{I}{m}\varrho^2\mathrm{ch}^2\frac{r}{R} & \frac{I}{m}\varrho^2\sin^2\varepsilon\,\mathrm{ch}\frac{r}{R} & \frac{I}{m}\varrho^2\cos^2\varepsilon\,\mathrm{ch}\frac{r}{R} \\ \frac{I}{m}\varrho^2\sin^2\varepsilon\,\mathrm{ch}\frac{r}{R} & \frac{I}{m}\varrho^2\sin^2\varepsilon & 0 \\ \frac{I}{m}\varrho^2\cos^2\varepsilon\,\mathrm{ch}\frac{r}{R} & 0 & \frac{I}{m}\varrho^2\cos^2\varepsilon \end{bmatrix},$$

$$K_2^{-1} = \begin{bmatrix} \frac{1}{R^2\mathrm{sh}^2\frac{r}{R}} & -\frac{\mathrm{ch}\frac{r}{R}}{R^2\mathrm{sh}^2\frac{r}{R}} & -\frac{\mathrm{ch}\frac{r}{R}}{R^2\mathrm{sh}^2\frac{r}{R}} \\ -\frac{\mathrm{ch}\frac{r}{R}}{R^2\mathrm{sh}^2\frac{r}{R}} & \frac{m}{I}\frac{1}{\varrho^2\sin^2\varepsilon} + \frac{1}{R^2}\mathrm{cth}^2\frac{r}{R} & \frac{1}{R^2}\mathrm{cth}^2\frac{r}{R} \\ -\frac{\mathrm{ch}\frac{r}{R}}{R^2\mathrm{sh}^2\frac{r}{R}} & \frac{1}{R^2}\mathrm{cth}^2\frac{r}{R} & \frac{m}{I}\frac{1}{\varrho^2\cos^2\varepsilon} + \frac{1}{R^2}\mathrm{cth}^2\frac{r}{R} \end{bmatrix}.$$

And obviously the other blocks are identical with those from the spherical geometry. The metrical scalar densities are given as follows:

$$G = \det\left[G_{ij}\right] = R^2\left(\frac{I}{m}\right)^4 \varrho^4\sin^2\varepsilon\cos^2\varepsilon\ \mathrm{sh}^2\frac{r}{R},$$

$$\sqrt{|G|} = R\left(\frac{I}{m}\right)^2 \varrho^2\left|\sin\varepsilon\cos\varepsilon\right|\mathrm{sh}\frac{r}{R}.$$

There are also no essential changes with the integration procedure, reasonable deformative potentials models, and the action-angle variables. Just the trigonometric functions of r/R replaced by the hyperbolic ones.

4.3 Toroidal case

Let us now consider another interesting two-dimensional model, namely, the motion of structured material points over the toroidal manifold, so this time over the Riemann space of not constant curvature. Obviously, we do not mean the torus $\mathbb{R}^2/\mathbb{Z}^2$, i.e., the quotient of $\mathbb{R}^2$ with respect to the "crystal lattice" $\mathbb{Z}^2$; this would be flat, although topologically not trivial, space. Our torus is a "tyre", or rather "inner tube", injected into $\mathbb{R}^3$ and endowed with the metric tensor induced from $\mathbb{R}^3$ (from its flat Euclidean structure).

Let R denote the "small" radius, i.e., the radius of circles obtained from transversal orthogonal cross-sections of the tube by planes. The "large" radius, i.e., the radius of the centrally placed inner-tube circle will be denoted by L. The parametric description of the torus is given as follows:

$$\begin{aligned} x &= (L + R\cos\vartheta)\cos\varphi, \\ y &= (L + R\cos\vartheta)\sin\varphi, \\ z &= R\sin\vartheta, \end{aligned}$$

where φ is the angle measured along the central inner circle ("along-tyre" angle) and ϑ denotes the polar angle of cross-section circles ("around-tyre" angle) measured from the external circle (maximally remote from the "tyre" centre). Both φ and ϑ run over the usual angular range $[0, 2\pi]$ with its obvious non-uniqueness. One can easily show that such a torus, denoted by $T^2(0, L, R)$, consists of points $(x, y, z) \in \mathbb{R}^3$ satisfying the fourth-degree algebraic equation

$$(x^2 + y^2 + z^2 + L^2 - R^2)^2 - 4L^2(x^2 + y^2) = 0.$$

Therefore, it is an algebraic manifold of fourth-degree because it is impossible to lower the degree of the polynomial on the left-hand side.

After the easy calculation one obtains that the metric element induced on the surface $T^2(0, L, R)$ has the form

$$ds^2 = R^2 d\vartheta^2 + (L + R\cos\vartheta)^2\, d\varphi^2.$$

Introducing as previously the geodetic length along the "around-tyre" circles, i.e.,

$$r = R\vartheta \in [0, 2\pi R]\,,$$

we can write that

$$ds^2 = dr^2 + \left(L + R\cos\frac{r}{R}\right)^2 d\varphi^2.$$

Therefore, the kinetic energy of the structure-less material point is given by the expression

$$\begin{aligned} T_{\mathrm{tr}} &= \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + \left(L + R\cos\frac{r}{R}\right)^2\left(\frac{d\varphi}{dt}\right)^2\right) \\ &= \frac{m}{2}\left(R^2\left(\frac{d\vartheta}{dt}\right)^2 + (L + R\cos\vartheta)^2\left(\frac{d\varphi}{dt}\right)^2\right). \end{aligned}$$

In analogy to spherical and pseudo-spherical formulas it is also convenient to use the form

$$T_{\rm tr} = \frac{mR^2}{2}\left(\left(\frac{d\vartheta}{dt}\right)^2 + \left(\frac{L}{R} + \cos\vartheta\right)^2\left(\frac{d\varphi}{dt}\right)^2\right).$$

It is seen that again φ is the cyclic variable in $T_{\rm tr}$. The along-φ rotations, i.e., $\varphi \mapsto \varphi + \alpha$, are isometries, thus, $\partial/\partial\varphi$ is the Killing vector field.

As previously, the coordinates on $T^2(0, L, R)$ will be ordered as (ϑ, φ) or (r, φ), thus,

$$\begin{aligned} [g_{ij}] &= \begin{bmatrix} R^2 & 0 \\ 0 & (L + R\cos\vartheta)^2 \end{bmatrix}, \\ \left[g^{ij}\right] &= \begin{bmatrix} \frac{1}{R^2} & 0 \\ 0 & \frac{1}{(L+R\cos\vartheta)^2} \end{bmatrix}, \end{aligned}$$

and the density of Riemannian volume (area) is given as follows:

$$\sqrt{|g|} = R\,(L + R\cos\vartheta)\,.$$

Separable potentials have the form

$$V(\vartheta, \varphi) = V_\vartheta(\vartheta) + \frac{V_\varphi(\varphi)}{(L + R\cos\vartheta)^2}.$$

The simplest and geometrically interesting is the class of models invariant under φ-rotations, i.e., $V_\varphi = 0$. The angle φ is then a cyclic variable. Unfortunately, even the simplest case, i.e., the geodetic model $V = 0$, is technically rather complicated and leads to elliptic integrals (after an appropriate change of coordinates).

One can easily show that the only not vanishing Christoffel symbols are as follows:

$$\Gamma^{\varphi}{}_{\varphi\vartheta} = \Gamma^{\varphi}{}_{\vartheta\varphi} = -\frac{R\sin\vartheta}{L + R\cos\vartheta}, \qquad \Gamma^{\vartheta}{}_{\varphi\varphi} = \left(\frac{L}{R} + \cos\vartheta\right)\sin\vartheta.$$

As expected, the most natural choice of the auxiliary field of frames E is one suited to coordinate lines, i.e.,

$$E_\vartheta = \frac{1}{R}\frac{\partial}{\partial\vartheta} = \frac{1}{R}\mathcal{E}_\vartheta, \qquad E_\varphi = \frac{1}{L + R\cos\vartheta}\frac{\partial}{\partial\varphi} = \frac{1}{L + R\cos\vartheta}\mathcal{E}_\varphi.$$

It is evidently orthogonal because

$$g_{\vartheta\varphi} = g_{\varphi\vartheta} = 0, \qquad g_{\vartheta\vartheta} = R^2, \qquad g_{\varphi\varphi} = (L + R\cos\vartheta)^2.$$

Let us now take into account internal degrees of freedom. We begin with the infinitesimal gyroscope. The angular velocities are then analytically expressed by the expressions

$$\omega_{\rm rl} = \frac{d\psi}{dt}, \qquad \omega = \omega_{\rm rl} + \omega_{\rm dr} = \frac{d\psi}{dt} + \sin\vartheta \frac{d\varphi}{dt}.$$

Therefore, the kinetic energy of the infinitesimal rotator is given as follows:

$$\begin{aligned} T &= T_{\rm tr} + T_{\rm int} \\ &= \frac{m}{2}\left(R^2\left(\frac{d\vartheta}{dt}\right)^2 + (L + R\cos\vartheta)^2\left(\frac{d\varphi}{dt}\right)^2\right) + \frac{I}{2}\left(\frac{d\psi}{dt} + \sin\vartheta\frac{d\varphi}{dt}\right)^2. \end{aligned}$$

If we write it in the previous form, i.e.,

$$T = \frac{m}{2} G_{ij}(q) \frac{dq^i}{dt}\frac{dq^j}{dt},$$

with the ordering

$$(q^1, q^2, q^3) = (\vartheta, \varphi, \psi),$$

then

$$[G_{ij}] = \begin{bmatrix} R^2 & 0 & 0 \\ 0 & (L + R\cos\vartheta)^2 + \frac{I}{m}\sin^2\vartheta & \frac{I}{m}\sin\vartheta \\ 0 & \frac{I}{m}\sin\vartheta & \frac{I}{m} \end{bmatrix}.$$

The inverse metric underlying the geodetic Hamiltonian is given by the expression

$$\left[G^{ij}\right] = \begin{bmatrix} \frac{1}{R^2} & 0 & 0 \\ 0 & \frac{1}{(L+R\cos\vartheta)^2} & -\frac{\sin\vartheta}{(L+R\cos\vartheta)^2} \\ 0 & -\frac{\sin\vartheta}{(L+R\cos\vartheta)^2} & \frac{m}{I} + \frac{\sin^2\vartheta}{(L+R\cos\vartheta)^2} \end{bmatrix}.$$

Up to the I-dependent normalization, the Riemannian density function is like that for the structure-less material point, i.e.,

$$\sqrt{|G|} = \sqrt{\frac{I}{m}}\, R\,(L + R\cos\vartheta)\,.$$

Just as in the spherical and pseudospherical spaces, the metric G is not diagonal in natural coordinates, therefore, without rather complicated analysis it would be difficult to fix the class of potentials compatible with the separation of variables procedure. And again evidently integrable models are isotropic ones when the variables φ, ψ are cyclic, i.e.,

$$H = \mathcal{T} + V(\vartheta) = \frac{1}{2} G^{ij} p_i p_j + V(\vartheta).$$

Obviously, the phase space coordinates are ordered in the usual way, i.e.,

$$\left(q^1, q^2, q^3; p_1, p_2, p_3\right) = \left(\vartheta, \varphi, \psi; p_\vartheta, p_\varphi, p_\psi\right).$$

For the action variables J_φ, J_ψ we obtain that

$$J_\varphi = \oint p_\varphi d\varphi = \int_0^{2\pi} \ell d\varphi = 2\pi\ell, \tag{4.54}$$

$$J_\psi = \oint p_\psi d\psi = \int_0^{2\pi} s d\psi = 2\pi s, \tag{4.55}$$

with the same meaning of symbols as previously and

$$S_0(\vartheta, \varphi, \psi; E, \ell, s) = S_\vartheta(\vartheta, E) + \ell\varphi + s\psi.$$

The true dynamics is contained in J_ϑ which after the substitution of (4.54) and (4.55) into

$$\oint p_\varphi d\vartheta = \oint \frac{dS_\vartheta}{d\vartheta} d\vartheta$$

becomes as follows:

$$J_\vartheta = R \oint \sqrt{2m\left(E - V(\vartheta)\right) - \frac{\left(J_\varphi - J_\psi \sin\vartheta\right)^2}{4\pi^2 \left(L + R\cos\vartheta\right)^2} + \frac{m}{I} \frac{J_\psi^2}{4\pi^2}} d\vartheta.$$

Assuming some particular shape of $V(\vartheta)$, "performing" the integration, and solving the result with respect to E, one obtains the final formula:

$$E = \mathcal{H}\left(J_\vartheta, J_\varphi, J_\psi\right).$$

No degeneracy is a priori seen without performing the integration, even for the geodetic models, i.e., when $V = 0$. An elementary integrability (if any!) may be suspected, of course, only for $V(\vartheta)$ built in a simple way from $\sin\vartheta, \cos\vartheta$-expressions.

Let us now quote some formulas concerning an infinitesimal deformable top moving over the toroidal surface. We restrict ourselves to the doubly isotropic dynamical models of internal degrees of freedom and use the two-polar decomposition of the matrix φ. All symbols are just as in the spherical and pseudo-spherical case, in particular, generalized coordinates q^i, $i = \overline{1,6}$, are ordered as follows:

$$\vartheta, \varphi, \gamma, \delta, x, y.$$

Angular velocities χ, θ of the two-polar decomposition are given by the expressions

$$\chi = \chi_{\mathrm{rl}} + \chi_{\mathrm{dr}} = \frac{d\alpha}{dt} + \sin\vartheta \frac{d\varphi}{dt}, \qquad \theta = \frac{d\beta}{dt},$$

where obviously α, β are the primary angles of the decomposition, i.e.,

$$\alpha = \frac{1}{2}(\gamma + \delta), \qquad \beta = \frac{1}{2}(\gamma - \delta).$$

The general formula (3.44) implies then that

$$T = T_{\mathrm{tr}} + T_{\mathrm{int}} = \frac{m}{2} G_{ij}(q) \frac{dq^i}{dt}\frac{dq^j}{dt},$$

where with the above ordering of generalized coordinates the matrix $[G_{ij}]$ consists of three distinguished blocks M_1, M_2, M_3, and namely,

$$M_1 = \left[R^2\right],$$

$$M_2 = \begin{bmatrix} (L + R\cos\vartheta)^2 + \frac{I}{m}\left(x^2 + y^2\right)\sin^2\vartheta & \frac{I}{m}x^2 \sin\vartheta & \frac{I}{m}y^2\sin^2\vartheta \\ \frac{I}{m}x^2\sin\vartheta & \frac{I}{m}x^2 & 0 \\ \frac{I}{m}y^2\sin^2\vartheta & 0 & \frac{I}{m}y^2 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} \frac{I}{m} & 0 \\ 0 & \frac{I}{m} \end{bmatrix} = \frac{I}{m} I_2.$$

The Riemann density has the form

$$\sqrt{|G|} = R\left(\frac{I}{m}\right)^2 |xy| \left(L + R\cos\vartheta\right).$$

The inverse matrix $\left[G^{ij}\right]$ consists of the blocks

$$M_1{}^{-1} = \left[\frac{1}{R^2}\right],$$

$$M_2{}^{-1} = \begin{bmatrix} \frac{1}{(L+R\cos\vartheta)^2} & -\frac{\sin\vartheta}{(L+R\cos\vartheta)^2} & -\frac{\sin\vartheta}{(L+R\cos\vartheta)^2} \\ -\frac{\sin\vartheta}{(L+R\cos\vartheta)^2} & \frac{m}{I}\frac{1}{x^2} + \frac{\sin^2\vartheta}{(L+R\cos\vartheta)^2} & \frac{\sin^2\vartheta}{(L+R\cos\vartheta)^2} \\ -\frac{\sin\vartheta}{(L+R\cos\vartheta)^2} & \frac{\sin^2\vartheta}{(L+R\cos\vartheta)^2} & \frac{m}{I}\frac{1}{y^2} + \frac{\sin^2\vartheta}{(L+R\cos\vartheta)^2} \end{bmatrix},$$

$$M_3{}^{-1} = \begin{bmatrix} \frac{m}{I} & 0 \\ 0 & \frac{m}{I} \end{bmatrix} = \frac{m}{I} I_2.$$

As in all previous examples, the G-non-orthogonality of the natural coordinates is a serious difficulty in analysis of integrability problems. And just as previously, one can do something when the angles φ, α, β are cyclic variables, i.e., when V depends only on ϑ and deformations invariants x, y. Namely, the system is separable for Hamiltonians of the form

$$H = \mathcal{T} + V_\vartheta(\vartheta) + V_x(x) + V_y(y). \tag{4.56}$$

When V_x, V_y are chosen so that to prevent an unlimited extension or contraction of the body, then they are capable to encode something like the dynamics of nonlinear elastic vibrations. This is even much more true when we use the polar coordinates (ϱ, ε), i.e., (4.33), on the plane of deformation invariants:

$$x = \varrho \sin\varepsilon, \qquad y = \varrho\cos\varepsilon.$$

In these coordinates the matrix $[G_{ij}]$ consists again of three blocks K_1, K_2, K_3, where $K_1 = M_1$, K_2 is just M_2 with formally substituted (4.33), and K_3 has the form

$$K_3 = \begin{bmatrix} \frac{I}{m} & 0 \\ 0 & \frac{I}{m}\varrho^2 \end{bmatrix} = \frac{I}{m}\begin{bmatrix} 1 & 0 \\ 0 & \varrho^2 \end{bmatrix}.$$

Just as in the spherical and pseudospherical case the problem is separable for deformation potentials (4.36):

$$V(\varrho,\varepsilon) = V_\varrho(\varrho) + \frac{V_\varepsilon(\varepsilon)}{\varrho^2},$$

i.e., for the total potentials

$$V(\vartheta,\varrho,\varepsilon) = V_\vartheta(\vartheta) + V_\varrho(\varrho) + \frac{V_\varepsilon(\varepsilon)}{\varrho^2}.$$

In particular, (4.41) is very interesting from the point of view of nonlinear elasticity.

The general procedure of constructing the action J-variables and discussing degeneracy problems is just like in the spherical and pseudospherical cases. Let us write down some explicit formulas.

According to our assumption about independence of V of the angles φ, α, β we have the following forms of the reduced action S_0:

$$\begin{aligned} S_0 &= S_\vartheta(\vartheta) + S_x(x) + S_y(y) + \ell\varphi + s\alpha + j\beta \\ &= S_\vartheta(\vartheta) + S_\varrho(\varrho) + S_\varepsilon(\varepsilon) + \ell\varphi + s\alpha + j\beta, \end{aligned}$$

depending on whether we use the (x, y)- or (ϱ, ε)-variables in the plane of deformations invariants. Therefore, just as previously,

$$\begin{aligned} J_\alpha &= \oint p_\alpha d\alpha = \int_0^{2\pi} s d\alpha = 2\pi s, \\ J_\beta &= \oint p_\beta d\beta = \int_0^{2\pi} j d\beta = 2\pi j, \\ J_\varphi &= \oint p_\varphi d\varphi = \int_0^{2\pi} \ell d\varphi = 2\pi \ell, \end{aligned}$$

where s, j, ℓ denote respectively the constant fixed values of spin, vorticity, and "orbital" angular momenta.

If Hamiltonian has the form (4.56), then the separation of variables procedure tells us that

$$J_x = \oint \sqrt{2I\left(C_x - V_x(x)\right) - \frac{\left(J_\alpha + J_\beta\right)^2}{16\pi^2 x^2}} dx, \tag{4.57}$$

$$J_y = \oint \sqrt{2I\left(C_y - V_y(y)\right) - \frac{\left(J_\alpha - J_\beta\right)^2}{16\pi^2 y^2}} dy, \tag{4.58}$$

$$J_\vartheta = R \oint \sqrt{2m\left(E - C_x - C_y - V_\vartheta(\vartheta)\right) - \frac{\left(J_\varphi - J_\alpha \sin\vartheta\right)^2}{4\pi^2\left(L + R\cos\vartheta\right)^2}} d\vartheta. \tag{4.59}$$

And again the procedure is like previously. When some explicit forms of V_x, V_y, V_ϑ are assumed, the integrals (4.57), (4.58) are "calculated", and then C_x, C_y are expressed as functions of J_x, J_α, J_β, i.e.,

$$C_x = C_x\left(J_x, J_\alpha + J_\beta\right), \qquad C_y = C_y\left(J_y, J_\alpha - J_\beta\right).$$

Substituting this to (4.59) and "calculating" the integral, we express J_ϑ through E, J_α, J_β, J_φ, J_x, J_y. And finally, solving this expression with respect to E, we determine E as a function of action variables, i.e.,

$$E = \mathcal{H}\left(J_\vartheta, J_\varphi, J_\alpha, J_\beta, J_x, J_y\right).$$

And unfortunately again without explicit calculations nothing may be decided a priori concerning even some partial degeneracy.

Chapter 5

Three-dimensional problems

Now we are going to discuss the motion of structured material points in three dimensions. We concentrate on the very interesting special case of the Einstein Universe, i.e., compact constant-curvature space metrically diffeomorphic with the three-dimensional sphere $S^3(0,R) \subset \mathbb{R}^4$. This three-dimensional model is quite different from the previously studied two-dimensional ones. The point is that the spheres $S^3(0,R)$ are parallelizable, moreover, $S^3(0,1)$ may be identified in a natural way with the group SU(2). This means that the most natural choice of the auxiliary reference frame E is that of basic generators of left or right regular translations on SU(2), i.e., respectively, the basic system of right- or left-invariant vector fields on SU(2). This frame may be easily chosen orthonormal. And all this means that the principal bundle of orthonormal frames over SU(2), $FM(\mathrm{SU}(2), g)$ may be naturally identified with the Cartesian product $\mathrm{SU}(2) \times \mathrm{SO}(3,\mathbb{R})$. The metric g here is obtained by the natural restriction of the Euclidean metric in $\mathbb{R}^4$ to $S^3(0,R)$ and in the SU(2)-terms it is identical up to a constant factor with the Killing-Cartan metric on SU(2). The latter is invariant under left and right regular translations on the group SU(2).

Before going any further we return temporarily to the more general n-dimensional formulation. This makes our description more lucid. So, we return to the general formula (3.32) for the co-moving affine velocity $\widehat{\Omega}$, i.e,

$$\widehat{\Omega}^B{}_A = \varphi^{-1B}{}_F \Gamma^F{}_{DC} \varphi^D{}_A \varphi^C{}_E \widehat{V}^E + \varphi^{-1B}{}_C \frac{d\varphi^C{}_A}{dt}. \tag{5.1}$$

Let us remind that $\widehat{\Omega}$ is defined by the formula

$$\frac{De_A}{Dt} = e_B \widehat{\Omega}^B{}_A,$$

i.e.,

$$\widehat{\Omega}^B{}_A = \left\langle e^B, \frac{De_A}{Dt} \right\rangle = e^B{}_i \frac{De^i{}_A}{Dt}.$$

When the internal motion is gyroscopic, i.e., $\varphi \in \mathrm{SO}(n, \mathbb{R})$, then obviously $\widehat{\Omega}$ is skew-symmetric in the Kronecker-delta sense, i.e.,

$$\widehat{\Omega}^B{}_A = -\widehat{\Omega}_A{}^B = -\delta_{AC}\delta^{BD}\widehat{\Omega}^C{}_D.$$

In the above formula (5.1) $\widehat{V}^A$ denote the co-moving components of the translational velocity, i.e.,

$$\widehat{V}^A = e^A{}_i V^i = e^A{}_i \frac{dx^i}{dt}.$$

And we shall use the abbreviations (3.38), i.e.,

$$\widehat{\Omega}_{\mathrm{dr}}{}^A{}_B = \varphi^{-1A}{}_F \Gamma^F{}_{DC} \varphi^D{}_B \varphi^C{}_E \widehat{V}^E, \tag{5.2}$$

$$\widehat{\Omega}_{\mathrm{rl}}{}^A{}_B = \varphi^{-1A}{}_C \frac{d\varphi^C{}_B}{dt}, \qquad \Omega_{\mathrm{rl}}{}^A{}_B = \frac{d\varphi^A{}_C}{dt} \varphi^{-1C}{}_B, \tag{5.3}$$

thus,

$$\widehat{\Omega} = \widehat{\Omega}_{\mathrm{dr}} + \widehat{\Omega}_{\mathrm{rl}} = \widehat{\Omega}_{\mathrm{dr}} + \varphi^{-1}\Omega_{\mathrm{rl}}\varphi.$$

If, as we always assume, the auxiliary frame E is g-orthonormal and Γ is the g-Levi-Civita connection, or more generally it is some Riemann-Cartan connection, then the aholonomic components $\Gamma^A{}_{BC}$ are Kronecker-skew-symmetric in the first pair of indices, i.e.,

$$\Gamma^A{}_{BC} = -\Gamma_B{}^A{}_C = -\delta_{BK}\delta^{AL}\Gamma^K{}_{LC}.$$

It is not true for the holonomic components $\Gamma^i{}_{jk}$ because they are not g-skew-symmetric,

$$\Gamma^i{}_{jk} \neq -\Gamma_j{}^i{}_k = -g_{ja}g^{ib}\Gamma^a{}_{bk}.$$

Let us now assume that the "legs" E_A of the auxiliary field of frames E form a Lie algebra, i.e.,

$$[E_A, E_B] = C^K{}_{AB} E_K,$$

where $C^K{}_{AB}$ are some constants, just the structure constants of the algebra with respect to the basis $(\ldots, E_A, \ldots)$. Lie bracket of vector fields in the above formula is meant in the convention

$$[X, Y]^i = X^j Y^i{}_{,j} - Y^j X^i{}_{,j},$$

i.e., as the commutator of X, Y when the vector fields are identified with first-order differential operators:

$$Xf = X^i f_{,i}.$$

If some point $x_0 \in M$ is fixed, then the manifold M may be identified with some Lie group and x_0 with its identity. Without distinguished x_0 the manifold M is a homogeneous space of the corresponding Lie group with trivial isotropy groups. In the Abelian case with the $\mathbb{R}^n$-topology of M this is just the difference between the linear and affine spaces.

The algebraic Killing metric of the Lie algebra with structure constants $C^K{}_{LM}$ has with respect to the basis $(\ldots, E_A, \ldots)$ the components

$$\gamma_{AB} = C^K{}_{LA} C^L{}_{KB}.$$

The corresponding Killing tensor field on M is given in local coordinates as follows:

$$\gamma_{ij} = \gamma_{AB} E^A{}_i E^B{}_j,$$

i.e., in the absolute tensor notation we have that

$$\gamma = \gamma_{AB} E^A \otimes E^B.$$

Obviously, it is not singular if and only if the Lie algebra given by structure constants C is semi-simple. It has the elliptic signature, i.e., is essentially Riemannian when the underlying Lie group is compact. Then strictly speaking the metric is negatively definite, but changing its over-all sign we obtain the usual positive metric. The inverse algebraic metric γ^{AB} satisfies the condition

$$\gamma^{AC} \gamma_{CB} = \delta^A{}_B,$$

thus, the contravariant metric field on M is given as follows:

$$\gamma^{ij} = E^i{}_A E^j{}_B \gamma^{AB},$$

or in the index-free form:

$$\widetilde{\gamma} = \gamma^{AB} E_A \otimes E_B.$$

While $\gamma_{ij} E^i{}_A E^j{}_B = \gamma_{AB} = \text{const}$, then the metric tensors γ, g on M define practically "the same" geometries. And from now on we assume that γ and g just coincide at least up to a constant multiplier. Let us mention by the way that this multiplier does not affect the Levi-Civita connection.

Let $\Gamma[E]$ denote the teleparallelism affine connection (3.26), (3.27) built of E and $S[E]$ is its torsion tensor (3.28). It follows from (3.30), (3.32) that

$S[E]^A{}_{BC}$, i.e., the aholonomic coefficients of $S[E]$ with respect to E, are given as follows:

$$S\left[E\right]^A{}_{BC} = \frac{1}{2} C^A{}_{BC}.$$

We have just assumed that the metric tensors γ, g differ by an overall constant multiplier λ, thus,

$$\gamma_{AB} = \lambda \delta_{AB}, \qquad \gamma_{ij} = \lambda g_{ij}. \tag{5.4}$$

Their common Levi-Civita connection will be denoted by $\{\}$ or componentwisely by $\left\{ \begin{array}{c} i \\ jk \end{array} \right\}$. According to the assumed relationships between g, E, γ, the metric tensors γ, g are parallel under the connection $\Gamma[E]$ (i.e., the teleparallelism connection is metrical with respect to γ, g). Therefore, following (2.1),

$$\Gamma\left[E\right]^i{}_{jk} = \left\{ \begin{array}{c} i \\ jk \end{array} \right\} + S^i{}_{jk} + S_{jk}{}^i + S_{kj}{}^i,$$

i.e., in the aholonomic terms:

$$\Gamma\left[E\right]^K{}_{LM} = \left\{ \begin{array}{c} K \\ LM \end{array} \right\} + \frac{1}{2} C^K{}_{LM} + \frac{1}{2} C_{LM}{}^K - \frac{1}{2} C_M{}^K{}_L.$$

The shift of "small" holonomic indices is meant here in the sense of g_{ij}, whereas the shift of aholonomic "capital" indices is meant in the Kronecker-delta sense. But the E-aholonomic components of the E-teleparallelism connection evidently vanish, i.e., $\Gamma[E]^K{}_{LM} = 0$, so we obtain that

$$\left\{ \begin{array}{c} K \\ LM \end{array} \right\} = -\frac{1}{2} C^K{}_{LM} - \frac{1}{2} C_{LM}{}^K + \frac{1}{2} C_M{}^K{}_L.$$

In semi-simple Lie algebras the structure constants are skew-symmetric in the first pair of indices with respect to the Killing metric. As we have assumed the latter to be proportional to the Kronecker delta, then we finally obtain that

$$\left\{ \begin{array}{c} K \\ LM \end{array} \right\} = -\frac{1}{2} C_M{}^K{}_L = -\frac{1}{2} C^K{}_{LM}.$$

So, finally (5.2), (5.3) imply that

$$\begin{aligned} \widehat{\Omega}^A{}_B &= \widehat{\Omega}_{\mathrm{dr}}{}^A{}_B + \widehat{\Omega}_{\mathrm{rl}}{}^A{}_B \\ &= -\frac{1}{2} \varphi^{-1A}{}_F C^F{}_{DC} \varphi^D{}_B \varphi^C{}_E \widehat{V}^E + \varphi^{-1A}{}_C \frac{d\varphi^C{}_B}{dt}. \end{aligned}$$

It turns out that in certain formulas for kinetic energy it may be convenient to use other representations for the relative (internal) aholonomic velocity. Namely, from the point of view of $\mathrm{GL}(n,\mathbb{R})$-objects $\widehat{\Omega}_{\mathrm{rl}}$ is the "co-moving" affine/angular velocity. One can also use the "spatial" representation in $\mathbb{R}^n$, i.e.,

$$\Omega_{\mathrm{rl}}{}^{A}{}_{B} = \frac{d\varphi^{A}{}_{C}}{dt}\varphi^{-1C}{}_{B} = \varphi^{A}{}_{C}\widehat{\Omega}_{\mathrm{rl}}{}^{C}{}_{D}\varphi^{-1D}{}_{B}.$$

There are also some additional possibilities concerning the M-representation, i.e., literally understood spatial representation. Obviously, the most geometric one is that based on the e-injection, i.e.,

$$\begin{aligned}\Omega^{i}{}_{j} &= e^{i}{}_{A}\widehat{\Omega}^{A}{}_{B}e^{B}{}_{j} = \frac{De^{i}{}_{A}}{Dt}e^{A}{}_{j},\\ \Omega_{\mathrm{rl}}{}^{i}{}_{j} &= e^{i}{}_{A}\widehat{\Omega}_{\mathrm{rl}}{}^{A}{}_{B}e^{B}{}_{j}, \qquad \varphi^{i}{}_{j} = e^{i}{}_{A}\varphi^{A}{}_{B}e^{B}{}_{j},\end{aligned}$$

and so on. But one can also use the E-injection of $\mathbb{R}^n$ into the tangent spaces of M, i.e.,

$$\widetilde{\Omega}_{\mathrm{rl}}{}^{i}{}_{j} = E^{i}{}_{A}\widehat{\Omega}_{\mathrm{rl}}{}^{A}{}_{B}E^{B}{}_{j}, \qquad \widetilde{V}^{i} = E^{i}{}_{A}\widehat{V}^{A}, \qquad \widetilde{\varphi}^{i}{}_{j} = E^{i}{}_{A}\varphi^{A}{}_{B}E^{B}{}_{j},$$

and so on.

Let us now go back to the three-dimensional simple compact groups $\mathrm{SU}(2)$, $\mathrm{SO}(3,\mathbb{R})$. In the standard representation structure constants are given by the three-dimensional Ricci symbol:

$$C^{K}{}_{LM} = \varepsilon^{K}{}_{LM}, \qquad C_{KLM} = \varepsilon_{KLM}$$

(again the shift of indices is meant in the Kronecker sense).

Some analytical expressions must be quoted now. Lie algebra of $\mathrm{SU}(2)$ consists of trace-less anti-hermitian 2×2 matrices and it is customary to use the standard basis $a \in \mathrm{SU}(2)'$, $j = 1,2,3$, where

$$a_B = \frac{1}{2i}\sigma_B,$$

and σ_B denote the Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i\\ i & 0\end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0\\ 0 & -1\end{bmatrix}.$$

Using canonical coordinates of the first kind $\overline{k} \in \mathbb{R}^3$, we have the expressions for the $\mathrm{SU}(2)$-elements

$$u\left(\overline{k}\right) = \exp\left(k^{B}a_{B}\right) = I_2\cos\frac{k}{2} - i\sigma_B\frac{k^B}{k}\sin\frac{k}{2}, \tag{5.5}$$

where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the 2×2 identity matrix and

$$k = \sqrt{\overline{k} \cdot \overline{k}} = \sqrt{(k^1)^2 + (k^2)^2 + (k^3)^2}$$

is the Euclidean length of $\overline{k}$. The direction $\overline{k}/k \in S^2(0,1)$ of $\overline{k}$ runs over the total indicated range, whereas the modulus k changes between 0 and 2π. All vectors $\overline{k}$ with $k = 2\pi$ refer to the same group element $-I_2 \in \mathrm{SU}(2)$. For the values of k larger than 2π the formula (5.5) would repeat the former elements of SU(2). In this way $k = 0$ and $k = 2\pi$ are singularities of the above coordinate system. Obviously, a_J obey the standard commutation rules, i.e.,

$$[a_J, a_K] = \varepsilon^M{}_{JK} a_M.$$

The Lie algebra of $\mathrm{SO}(3,\mathbb{R})$, i.e., $\mathrm{SO}(3,\mathbb{R})'$, consists of real 3×3 skew-symmetric matrices. The standard basis is given by A_K, $K = 1, 2, 3$, where

$$(A_K)^L{}_M = -\varepsilon_K{}^L{}_M,$$

and the trivial (purely cosmetic) Kronecker shift of indices is meant. With this convention we have the same commutation rules, i.e.,

$$[A_J, A_K] = \varepsilon^M{}_{JK} A_M.$$

The exponential mapping

$$R\left(\overline{k}\right) = \exp\left(k^J A_J\right) \tag{5.6}$$

leads to the explicit form of finite rotation matrices $R\left(\overline{k}\right)$:

$$R\left(\overline{k}\right) \cdot \overline{x} = \cos k \cdot \overline{x} + (1 - \cos k)\left(\frac{\overline{k}}{k} \cdot \overline{x}\right)\frac{\overline{k}}{k} + \sin k \, \frac{\overline{k}}{k} \times \overline{x},$$

with the standard notation $\overline{a} \cdot \overline{b}$ and $\overline{a} \times \overline{b}$ respectively for the scalar and vector product in $\mathbb{R}^3$.

Now k runs over the range $[0, \pi]$ and antipodal points on the coordinate sphere $k = \pi$ in $\mathbb{R}^3$ are pairwise identified because

$$R(\pi\overline{n}) = R(-\pi\overline{n})$$

for any versor $\overline{n} \in \mathbb{R}^3$, $\overline{n} \cdot \overline{n} = 1$. In this way $\overline{k}$ is the usual rotation vector, k is the rotation angle (in radians), and $\overline{n} = \overline{k}/k$ is the rotation axis in the right-hand-screw sense, $R(\overline{k})\overline{k} = \overline{k}$. In particular, we have the intuitive formula

$$R(\overline{k})\overline{x} = \overline{x} + \overline{k} \times \overline{x} + \frac{1}{2!}\overline{k} \times (\overline{k} \times \overline{x}) + \cdots + \frac{1}{n!}\overline{k} \times (\overline{k} \times \cdots (\overline{k} \times \overline{x}) \ldots) + \cdots,$$

where the term with the $1/n!$-factor contains n copies of $\overline{k}$. The natural homomorphism

$$\mathrm{pr} : \mathrm{SU}(2) \to \mathrm{SO}(3, \mathbb{R})$$

is given by $v \mapsto R$, where

$$vu(\overline{k})v^{-1} = u(R\overline{k}).$$

Therefore,

$$\mathrm{pr}^{-1}\left(R(\overline{k})\right) = \left\{u(\overline{k}), u\left(-(2\pi - k)\frac{\overline{k}}{k}\right)\right\} = \left\{u(\overline{k}), -u(\overline{k})\right\}.$$

The formula (5.5) injects the group SU(2) into the real four-dimensional space spanned by matrices I_2, $-i\sigma_B$, $B = 1, 2, 3$. Obviously, this space may be identified with $\mathbb{R}^4$. It is seen that the sum of squared coefficients at the mentioned matrices equals one. This is just the mentioned identification between SU(2) and three-dimensional unit sphere $S^3(0, 1) \subset \mathbb{R}^4$.

The Lie-algebraic Killing metric built of the structure constants $\varepsilon^{A}{}_{BC}$ has the components

$$\gamma_{AB} = -2\delta_{AB},$$

therefore, in (5.4) we have that

$$\lambda = -2.$$

The metric field on SU(2) or $\mathrm{SO}(3, \mathbb{R})$ is analytically given as follows:

$$g_{ij} = \frac{4}{k^2}\sin^2\frac{k}{2}\delta_{ij} + \left(1 - \frac{4}{k^2}\sin^2\frac{k}{2}\right)\frac{k^i}{k}\frac{k^j}{k}, \qquad (5.7)$$

i.e., the corresponding arc element has the form

$$ds^2 = dk^2 + 4\sin^2\frac{k}{2}\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right), \qquad (5.8)$$

where k, ϑ, φ are spherical variables in the space of rotation vectors $\overline{k}$. The angular variables ϑ, φ parameterize the manifold of rotation versors $\overline{n}(\vartheta, \varphi) = \overline{k}/k$ and we can write that

$$ds^2 = dk^2 + 4\sin^2\frac{k}{2}d\overline{n}\cdot d\overline{n},$$

or using more sophisticated tensorial terms:

$$g = dk \otimes dk + 4\sin^2\frac{k}{2}\delta_{AB}dn^A \otimes dn^B.$$

The reference frame E may be chosen either as ${}^{l}E$ or ${}^{r}E$ respectively consisting of vector fields generating left or right translations on SU(2), SO(3, $\mathbb{R}$). The vector fields ${}^{l}E_A$, ${}^{r}E_A$ are respectively right- and left-invariant. Let ${}^{l}\widetilde{E}$, ${}^{r}\widetilde{E}$ be their dual co-frames. The covector fields (differential one-forms) ${}^{l}E^A$, ${}^{r}E^A$ are respectively right- and left-invariant.

After some calculations one obtains the analytical expressions in terms of coordinates $\overline{k}$:

$$
\begin{aligned}
{}^{l}E_A &= \frac{k}{2}\mathrm{ctg}\frac{k}{2}\frac{\partial}{\partial k^A} + \left(1 - \frac{k}{2}\mathrm{ctg}\frac{k}{2}\right)\frac{k_A}{k}\frac{k^J}{k}\frac{\partial}{\partial k^J} + \frac{1}{2}\varepsilon_{AM}{}^{J}k^M\frac{\partial}{\partial k^J},\\
{}^{r}E_A &= \frac{k}{2}\mathrm{ctg}\frac{k}{2}\frac{\partial}{\partial k^A} + \left(1 - \frac{k}{2}\mathrm{ctg}\frac{k}{2}\right)\frac{k_A}{k}\frac{k^J}{k}\frac{\partial}{\partial k^J} - \frac{1}{2}\varepsilon_{AM}{}^{J}k^M\frac{\partial}{\partial k^J},\\
{}^{l}E^A &= \frac{\sin k}{k}dk^A + \left(1 - \frac{\sin k}{k}\right)\frac{k^A}{k}\frac{k_B}{k}dk^B + \frac{2}{k^2}\sin^2\frac{k}{2}\ \varepsilon^{A}{}_{BC}k^B dk^C,\\
{}^{r}E^A &= \frac{\sin k}{k}dk^A + \left(1 - \frac{\sin k}{k}\right)\frac{k^A}{k}\frac{k_B}{k}dk^B - \frac{2}{k^2}\sin^2\frac{k}{2}\ \varepsilon^{A}{}_{BC}k^B dk^C.
\end{aligned}
$$

The vector fields

$$D_A := {}^{l}E_A - {}^{r}E_A = \varepsilon_{AB}{}^{C}k^B\frac{\partial}{\partial k^C}$$

are infinitesimal generators of inner automorphisms, i.e.,

$$u \mapsto vuv^{-1}.$$

It is convenient to separate explicitly the “radial” variable k and the angles ϑ, φ. To do this in a symmetric way, we use differential operators (vector fields) D_A which depend only on ϑ, φ and act only on these variables, i.e.,

$$D_A f = 0,$$

if f is a function of the variable k. Similarly, the redundant system of quantities $\overline{n} = \overline{k}/k$ are used. Obviously, n^A depend only on ϑ, φ, but not on k.

Then one can show that

$$
\begin{aligned}
{}^{l}E_A &= n_A\frac{\partial}{\partial k} - \frac{1}{2}\mathrm{ctg}\frac{k}{2}\ \varepsilon_{ABC}n^B D^C + \frac{1}{2}D_A,\\
{}^{r}E_A &= n_A\frac{\partial}{\partial k} - \frac{1}{2}\mathrm{ctg}\frac{k}{2}\ \varepsilon_{ABC}n^B D^C - \frac{1}{2}D_A,\\
{}^{l}E^A &= n^A dk + 2\sin^2\frac{k}{2}\ \varepsilon^{ABC}n_B dn_C + \sin k\ dn^A,\\
{}^{r}E^A &= n^A dk - 2\sin^2\frac{k}{2}\ \varepsilon^{ABC}n_B dn_C + \sin k\ dn^A,
\end{aligned}
$$

or in a brief, little symbolic way:

$$\begin{aligned}
{}^{l}\overline{E} &= \overline{n}\frac{\partial}{\partial k} - \frac{1}{2}\mathrm{ctg}\frac{k}{2}\ \overline{n}\times\overline{D} + \frac{1}{2}\overline{D},\\
{}^{r}\overline{E} &= \overline{n}\frac{\partial}{\partial k} - \frac{1}{2}\mathrm{ctg}\frac{k}{2}\ \overline{n}\times\overline{D} - \frac{1}{2}\overline{D},\\
{}^{l}\widetilde{E} &= \overline{n}dk + 2\sin^2\frac{k}{2}\ \overline{n}\times d\overline{n} + \sin k\ d\overline{n},\\
{}^{r}\widetilde{E} &= \overline{n}dk - 2\sin^2\frac{k}{2}\ \overline{n}\times d\overline{n} + \sin k\ d\overline{n}.
\end{aligned}$$

Obviously,

$$\langle dk, D_A\rangle = D_A k = 0, \qquad \left\langle dk, \frac{\partial}{\partial k}\right\rangle = 1,$$

$$\left\langle dn_A, \frac{\partial}{\partial k}\right\rangle = \frac{\partial n_A}{\partial k} = 0, \qquad \langle dn_A, D_B\rangle = D_B n_A = \varepsilon_{ABC}n^C.$$

Commutation relations between the above vector fields read as follows:

$$\left[{}^{l}E_A, {}^{l}E_B\right] = -\varepsilon_{AB}{}^{C}\ {}^{l}E_C, \qquad \left[{}^{r}E_A, {}^{r}E_B\right] = \varepsilon_{AB}{}^{C}\ {}^{r}E_C,$$

$$\left[{}^{l}E_A, {}^{r}E_B\right] = 0.$$

The vanishing of the last Lie bracket is due to the fact that the left and right group translations mutually commute.

The metric tensor g (see (5.7), (5.8)), i.e., the (-2)-multiple of the Killing tensor, may be written down as follows:

$$g = \delta_{AB}{}^{l}E^A \otimes {}^{l}E^B = \delta_{AB}{}^{r}E^A \otimes {}^{r}E^B.$$

Similarly, the contravariant inverse metric

$$g^{ij} = \frac{k^2}{4\sin^2\frac{k}{2}}\delta^{ij} + \left(1 - \frac{k^2}{4\sin^2\frac{k}{2}}\right)n^i n^j$$

may be expressed as follows:

$$\widetilde{g} = \delta^{AB}{}^{l}E_A \otimes {}^{l}E_B = \delta^{AB}{}^{r}E_A \otimes {}^{r}E_B,$$

or using the $k, \overline{n}$-variables:

$$\widetilde{g} = \frac{\partial}{\partial k}\otimes\frac{\partial}{\partial k} + \frac{1}{4\sin^2\frac{k}{2}}\delta^{AB}D_A\otimes D_B.$$

Let us now pass over from the standard unit sphere $S^3(0,1) \simeq \mathrm{SU}(2)$ to the sphere $S^3(0,R) \subset \mathbb{R}^4$ of the radius R. We introduce the new "radial" variable $r = Rk/2$, i.e., $k = 2r/R$. Obviously, the range of r is $[0, \pi R]$. The coordinate vector $\overline{k}$ is replaced by $\overline{r}$ with the length r and such that

$$\frac{\overline{r}}{r} = \frac{\overline{k}}{k} = \overline{n}.$$

Expressions for the auxiliary reference frames E and co-frames $\widetilde{E}$ take on the forms

$$
\begin{aligned}
{}^{l}E_A &= \frac{R}{2}{}^{l}E(R)_A, & {}^{r}E_A &= \frac{R}{2}{}^{r}E(R)_A,\\
{}^{l}E^A &= \frac{2}{R}{}^{l}E(R)^A, & {}^{r}E^A &= \frac{2}{R}{}^{r}E(R)^A,
\end{aligned}
$$

where

$$
\begin{aligned}
{}^{l}E(R)_A &= n_A\frac{\partial}{\partial r} - \frac{1}{R}\mathrm{ctg}\frac{r}{R}\varepsilon_{ABC}n^B D^C + \frac{1}{R}D_A, && (5.9)\\
{}^{r}E(R)_A &= n_A\frac{\partial}{\partial r} - \frac{1}{R}\mathrm{ctg}\frac{r}{R}\varepsilon_{ABC}n^B D^C - \frac{1}{R}D_A, && (5.10)\\
{}^{l}E(R)^A &= n^A dr + R\sin^2\frac{r}{R}\varepsilon^{ABC}n_B dn_C + \frac{R}{2}\sin\frac{2r}{R}dn^A, && (5.11)\\
{}^{r}E(R)^A &= n^A dr - R\sin^2\frac{r}{R}\varepsilon^{ABC}n_B dn_C + \frac{R}{2}\sin\frac{2r}{R}dn^A. && (5.12)
\end{aligned}
$$

The basic Lie brackets are as follows:

$$
\begin{aligned}
\left[{}^{l}E(R)_A, {}^{l}E(R)_B\right] &= -\frac{2}{R}\varepsilon_{AB}{}^{C\,l}E(R)_C, && (5.13)\\
\left[{}^{r}E(R)_A, {}^{r}E(R)_B\right] &= \frac{2}{R}\varepsilon_{AB}{}^{C\,r}E(R)_C, && (5.14)\\
\left[{}^{l}E(R)_A, {}^{r}E(R)_B\right] &= 0. && (5.15)
\end{aligned}
$$

Therefore, the corresponding structure constants are given as follows:

$${}^{l}C(R)^A{}_{BC} = -\frac{2}{R}\varepsilon^A{}_{BC}, \qquad {}^{r}C(R)^A{}_{BC} = \frac{2}{R}\varepsilon^A{}_{BC},$$

and the trivial (cosmetic) Kronecker shift of indices is meant.

The resulting metric

$$g(R) = \delta_{AB}{}^{l}E(R)^A \otimes {}^{l}E(R)^B = \delta_{AB}{}^{r}E(R)^A \otimes {}^{r}E(R)^B$$

has the coordinate form

$$g(R)_{ij} = \left(\frac{R}{r}\right)^2 \sin^2 \frac{r}{R}\delta_{ij} + \left(1 - \left(\frac{R}{r}\right)^2 \sin^2 \frac{r}{R}\right) n_i n_j,$$

i.e.,

$$\begin{aligned} ds^2 &= dr^2 + R^2 \sin^2 \frac{r}{R}\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) \\ &= dr^2 + R^2 \sin^2 \frac{r}{R} d\overline{n} \cdot d\overline{n}, \end{aligned} \tag{5.16}$$

or in more sophisticated terms:

$$g(R) = dr \otimes dr + R^2 \sin^2 \frac{r}{R}\delta_{AB} dn^A \otimes dn^B.$$

The algebraic Killing metric γ_{AB} is now given as follows:

$$\gamma_{AB} = -\frac{8}{R^2}\delta_{AB},$$

therefore,

$$\lambda = -\frac{8}{R^2}.$$

The above analytical formulas may be geometrically interpreted in such a way that the unit sphere in $\mathbb{R}^4$ is subject to the R-factor dilatation resulting in $S^3(0, R)$, i.e., the sphere of radius R and origin 0 in $\mathbb{R}^4$. The metric $g(R)$ may be obtained by restriction of the Kronecker metric in $\mathbb{R}^4$ to $S^3(0, R)$.

Parametrizing $S^3(0, R) \subset \mathbb{R}^4$ by (r, ϑ, φ)-coordinates, i.e.,

$$x^1 = R\sin\frac{r}{R}\sin\vartheta\cos\varphi, \qquad x^2 = R\sin\frac{r}{R}\sin\vartheta\sin\varphi,$$
$$x^3 = R\sin\frac{r}{R}\cos\vartheta, \qquad x^4 = R\cos\frac{r}{R},$$

and expressing

$$ds^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 + \left(dx^4\right)^2$$

through (r, ϑ, φ), one obtains exactly (5.16).

It is seen from (5.9), (5.10), (5.11), (5.12), (5.13), (5.14), and (5.15) that when the limit transition $R \to \infty$ is performed, then ${}^{l}E(R)_A$ and ${}^{r}E(R)_A$ become asymptotically commutative, then

$$\lim_{R\to\infty} {}^{l}E(R)_A = \lim_{R\to\infty} {}^{r}E(R)_A = \frac{\partial}{\partial x^A}.$$

Similarly,

$$\lim_{R\to\infty} {}^{l}E(R)^{A} = \lim_{R\to\infty} {}^{r}E(R)^{A} = dx^{A},$$
$$\lim_{R\to\infty} g(R)_{ij} = \delta_{ij},$$

i.e., as expected, one obtains the $\mathbb{R}^3$-relationships (flat space).

Now we make use of the peculiarity of the dimension three, i.e., the identification between skew-symmetric second-order tensors and axial vectors. Therefore, the matrices of angular velocity will be represented as follows:

$$\widehat{\Omega}(R)^{A}{}_{B} = -\varepsilon^{A}{}_{BC}\widehat{\Omega}(R)^{C}, \qquad \widehat{\Omega}_{\rm rl}{}^{A}{}_{B} = -\varepsilon^{A}{}_{BC}\widehat{\Omega}_{\rm rl}{}^{C},$$
$$\widehat{\Omega}_{\rm dr}(R)^{A}{}_{B} = -\varepsilon^{A}{}_{BC}\widehat{\Omega}_{\rm dr}(R)^{C}, \qquad \Omega_{\rm rl}{}^{A}{}_{B} = \frac{d\varphi^{A}{}_{C}}{dt}\varphi^{-1C}{}_{B} = -\varepsilon^{A}{}_{BC}\Omega_{\rm rl}{}^{C},$$

and conversely,

$$\widehat{\Omega}(R)^{A} = -\frac{1}{2}\varepsilon^{A}{}_{B}{}^{C}\widehat{\Omega}(R)^{B}{}_{C}, \qquad \widehat{\Omega}_{\rm rl}{}^{A} = -\frac{1}{2}\varepsilon^{A}{}_{B}{}^{C}\widehat{\Omega}_{\rm rl}{}^{B}{}_{C},$$
$$\widehat{\Omega}_{\rm dr}(R)^{A} = -\frac{1}{2}\varepsilon^{A}{}_{B}{}^{C}\widehat{\Omega}_{\rm dr}(R)^{B}{}_{C}, \qquad \Omega_{\rm rl}{}^{A} = -\frac{1}{2}\varepsilon^{A}{}_{B}{}^{C}\Omega_{\rm rl}{}^{B}{}_{C}.$$

As always, the capital indices are moved in the "cosmetic" Kronecker sense. There is no radius label R at $\widehat{\Omega}_{\rm rl}$, $\Omega_{\rm rl}$ because these quantities are independent of R.

After some calculations one obtains the expressions for the basic aholonomic velocities ("angular velocities") for the motion on the sphere $S^3(0,R)$:

$${}^{l}\Omega_{\rm dr}(R)^{A} = {}^{l}E^{A}{}_{i}(R;\overline{r})\frac{dr^{i}}{dt}, \qquad {}^{r}\Omega_{\rm dr}(R)^{A} = {}^{r}E^{A}{}_{i}(R;\overline{r})\frac{dr^{i}}{dt},$$
$${}^{l}\Omega_{\rm rl}{}^{A} = {}^{l}E^{A}{}_{i}(\overline{\varkappa})\frac{d\varkappa^{i}}{dt}, \qquad {}^{r}\Omega_{\rm rl}{}^{A} = {}^{r}E^{A}{}_{i}(\overline{\varkappa})\frac{d\varkappa^{i}}{dt},$$

where $\overline{\varkappa}$ is the rotation vector on $\mathrm{SO}(3,\mathbb{R})$ which parameterizes the manifold of internal degrees of freedom in the sense of (5.6) with $\overline{\varkappa}$ substituted instead of $\overline{k}$:

$$\mathrm{SO}(3,\mathbb{R}) \ni \varphi(\overline{\varkappa}) = \exp\left(\varkappa^{J}A_{J}\right).$$

Denoting the canonical momenta conjugate to r^i, $\varkappa^i$ respectively by p_i, π_i, we have the expressions for the aholonomic momenta (canonical angular momenta) conjugate to $\Omega_{\rm dr}(R)^{A}$, $\Omega_{\rm rl}{}^{A}$:

$${}^{l}S_{\rm dr}(R)_{A} = {}^{l}E^{i}{}_{A}(R;\overline{r})p_{i}, \qquad {}^{r}S_{\rm dr}(R)_{A} = {}^{r}E^{i}{}_{A}(R;\overline{r})p_{i},$$
$${}^{l}S_{{\rm rl}A} = {}^{l}E^{i}{}_{A}(\overline{\varkappa})\pi_{i}, \qquad {}^{r}S_{{\rm rl}A} = {}^{r}E^{i}{}_{A}(\overline{\varkappa})\pi_{i}.$$

Their Poisson brackets are given as follows:

$$\begin{aligned}
\left\{{}^{l}S_{\mathrm{dr}}(R)_A, {}^{l}S_{\mathrm{dr}}(R)_B\right\} &= \frac{2}{R}\varepsilon_{AB}{}^{C}\ {}^{l}S_{\mathrm{dr}}(R)_C,\\
\left\{{}^{r}S_{\mathrm{dr}}(R)_A, {}^{r}S_{\mathrm{dr}}(R)_B\right\} &= -\frac{2}{R}\varepsilon_{AB}{}^{C}\ {}^{r}S_{\mathrm{dr}}(R)_C,\\
\left\{{}^{l}S_{\mathrm{dr}}(R)_A, {}^{r}S_{\mathrm{dr}}(R)_B\right\} &= 0,\\
\left\{{}^{l}S_{\mathrm{rl}A}, {}^{l}S_{\mathrm{rl}B}\right\} &= \varepsilon_{AB}{}^{C}\ {}^{l}S_{\mathrm{rl}C},\\
\left\{{}^{r}S_{\mathrm{rl}A}, {}^{r}S_{\mathrm{rl}B}\right\} &= -\varepsilon_{AB}{}^{C}\ {}^{r}S_{\mathrm{rl}C},\\
\left\{{}^{l}S_{\mathrm{rl}A}, {}^{r}S_{\mathrm{rl}B}\right\} &= 0.
\end{aligned}$$

The drive and relative quantities are mutually in involution, e.g.,

$$\left\{{}^{l}S_{\mathrm{dr}}(R)_A, {}^{l}S_{\mathrm{rl}B}\right\} = 0,$$

and so on. After some calculations we obtain the expression for the kinetic energy

$$\begin{aligned}
T &= \frac{1}{2}\left(m + \frac{I}{R^2}\right)\delta_{AB}{}^{l}\Omega(R)^{A\,l}\Omega(R)^{B}\\
&\quad -\frac{I}{R}\delta_{AB}{}^{l}\Omega_{\mathrm{rl}}{}^{A}\ {}^{l}\Omega(R)^{B} + \frac{I}{2}\delta_{AB}{}^{l}\Omega_{\mathrm{rl}}{}^{A}\ {}^{l}\Omega_{\mathrm{rl}}{}^{B}.
\end{aligned}$$

There are a few interesting features of this formula, namely, existing of the mass-modifying term I/R^2 in the purely translational part and the characteristic gyroscopic term of coupling between two angular velocities.

The most convenient procedure of deriving equations of motion is one based on the Hamiltonian formalism and Poisson brackets. We do almost exclusively with the potential models when Lagrangian has the form

$$L = T - V\left(\overline{r}, \overline{\varkappa}\right).$$

The Legendre transformation expressed in aholonomic terms is as follows:

$$\begin{aligned}
{}^{l}S(R)_A &= \frac{\partial L}{\partial^{l}\Omega(R)^A} = \frac{\partial T}{\partial^{l}\Omega(R)^A},\\
{}^{l}S_{\mathrm{rl}A} &= \frac{\partial L}{\partial^{l}\Omega_{\mathrm{rl}}{}^{A}} = \frac{\partial T}{\partial^{l}\Omega_{\mathrm{rl}}{}^{A}},
\end{aligned}$$

i.e., explicitly,

$$\begin{aligned}
{}^{l}S(R)_A &= \left(m + \frac{I}{R^2}\right){}^{l}\Omega(R)_A - \frac{I}{R}{}^{l}\Omega_{\mathrm{rl}A},\\
{}^{l}S_{\mathrm{rl}A} &= -\frac{I}{R}{}^{l}\Omega(R)_A + I^{l}\Omega_{\mathrm{rl}A}.
\end{aligned}$$

The corresponding inverse rule reads

$$
\begin{aligned}
{}^{l}\Omega(R)^{A} &= \frac{1}{m}{}^{l}S(R)^{A} + \frac{1}{mR}{}^{l}S_{\mathrm{rl}}{}^{A},\\
{}^{l}\Omega_{\mathrm{rl}}{}^{A} &= \frac{1}{mR}{}^{l}S(R)^{A} + \frac{I+mR^{2}}{ImR^{2}}{}^{l}S_{\mathrm{rl}}{}^{A},
\end{aligned}
$$

where the raising and lowering of capital indices are meant in the "cosmetic" Kronecker-delta sense.

After substituting all this to the kinetic energy formula, we obtain the expression for the kinetic Hamiltonian

$$
\begin{aligned}
\mathcal{T} &= \frac{1}{2m}\delta^{AB}\,{}^{l}S(R)_{A}{}^{l}S(R)_{B} + \frac{1}{mR}\delta^{AB}\,{}^{l}S(R)_{A}{}^{l}S_{\mathrm{rl}B}\\
&\quad + \frac{I+mR^{2}}{2ImR^{2}}\delta^{AB}{}^{l}S_{\mathrm{rl}A}{}^{l}S_{\mathrm{rl}B}. \qquad (5.17)
\end{aligned}
$$

When some potential term is admitted, then the Hamiltonian has the form

$$H = \mathcal{T} + V(\overline{r}, \overline{\varkappa}).$$

For the purely geodetic models, when $V = 0$, the Poisson-bracket form of equations of motion, i.e.,

$$\frac{dF}{dt} = \{F, H\},$$

leads to the dynamical system for canonical angular momenta

$$
\begin{aligned}
\frac{d\,{}^{l}S(R)_{A}}{dt} &= \frac{2}{mR^{2}}\varepsilon_{A}{}^{BC}{}^{l}S_{\mathrm{rl}B}{}^{l}S(R)_{C}, \qquad (5.18)\\
\frac{d\,{}^{l}S_{\mathrm{rl}A}}{dt} &= \frac{1}{mR}\varepsilon_{A}{}^{BC}{}^{l}S(R)_{B}{}^{l}S_{\mathrm{rl}C}. \qquad (5.19)
\end{aligned}
$$

It is interesting that only the interference term in (5.17) contributes to these equations because the first and the third terms have vanishing Poisson brackets with ${}^{l}S(R)_{A}$, ${}^{l}S_{\mathrm{rl}A}$ (they are Casimir invariants). Using the three-dimensional vector notation in $\mathbb{R}^{3}$, we can write the above equations as follows:

$$
\begin{aligned}
\frac{d\,{}^{l}\overline{S(R)}}{dt} &= \frac{2}{mR^{2}}{}^{l}\overline{S_{\mathrm{rl}}} \times {}^{l}\overline{S(R)}, \qquad (5.20)\\
\frac{d\,{}^{l}\overline{S_{\mathrm{rl}}}}{dt} &= \frac{1}{mR}{}^{l}\overline{S(R)} \times {}^{l}\overline{S_{\mathrm{rl}}}, \qquad (5.21)
\end{aligned}
$$

where there is meant the standard vector product in $\mathbb{R}^{3}$.

The Poisson rules quoted above imply that the squared magnitudes of angular momenta, i.e.,

$$\begin{aligned} \left| {}^{l}S(R) \right|^{2} &= \delta^{AB\,l}S(R)_{A}{}^{l}S(R)_{B}, \\ \left| {}^{l}S_{\mathrm{rl}} \right|^{2} &= \delta^{AB\,l}S_{\mathrm{rl}A}{}^{l}S_{\mathrm{rl}B}, \end{aligned}$$

are constants of motion in virtue of the equations above (5.18), (5.19), (5.20), (5.21). And moreover, these equations imply that the $\mathbb{R}^{3}$-vector

$$\frac{R}{2}{}^{l}S(R)_{A} + {}^{l}S_{\mathrm{rl}A} \tag{5.22}$$

is a constant of motion. This implies in particular that the $\mathbb{R}^{3}$-scalar product of angular momenta

$$\delta^{AB}\ {}^{l}S(R)_{A}{}^{l}S_{\mathrm{rl}B}$$

and the angle between ${}^{l}\overline{S(R)}$, ${}^{l}\overline{S_{\mathrm{rl}}}$ are constants of motion functionally dependent on the previous ones.

In this way the six-dimensional dynamical system (5.18), (5.19), (5.20), (5.21) has five independent constants of motion and there is only one time-dependent degree of freedom in the $\left({}^{l}\overline{S(R)}, {}^{l}\overline{S}\right)$-space. The only essentially active degree of freedom is the orientation (polar angle) of the two-dimensional plane in $\mathbb{R}^{3}$ spanned by the vectors ${}^{l}\overline{S(R)}$, ${}^{l}\overline{S_{\mathrm{rl}}}$ and rotating about the axis given by the vector (5.22).

Remark: it has been told that the motion of vectors ${}^{l}\overline{S(R)}$, ${}^{l}\overline{S_{\mathrm{rl}}}$ is independent of the first and second terms in (5.17), in particular, of the inertial moment I. But of course these terms and the quantity I are relevant for the total description of motion. Namely, I appears in the Legendre transformation and its inverse. Therefore, the time evolution of configuration variables $(\overline{r}, \overline{\varkappa})$ depends explicitly on I and on all terms in the kinetic energy expression.

Bibliography

[1] R. Abraham and J. E. Marsden, *Foundations of Mechanics* (second ed.), The Benjamin-Cummings Publishing Company, London-Amsterdam-Sydney-Tokyo, 1978.

[2] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer Graduate Texts in Mathematics, vol. **60**, Springer-Verlag, New York, 1978.

[3] E. Binz, *Global Differential Geometric Methods in Elasticity and Hydrodynamics*, in: Differential Geometry, Group Representations and Quantization, Lecture Notes in Physics, vol. **379**, edited by J. D. Hennig, W. Lücke, and J. Tolar, Springer-Verlag, Berlin-Heidelberg, 1991.

[4] O. I. Bogoyavlensky, *Methods of Qualitative Theory of Dynamical Systems in Astrophysics and Gas Dynamics*, Springer, Berlin-Heidelberg-New York, 1985.

[5] A. Burov and D. P. Chevallier, *On the Variational Principle of Poincare, the Poincare-Chetayev Equations and the Dynamics of Affinely Deformable Bodies*, Cahier de C.E.R.M.I.C.S. **14**, Mai 1996.

[6] A. Burov and S. Ya. Stepanov, *On Geometry of Masses in Dynamics of Deformable Bodies*, in: *Problems of Investigation on Stability and Stabilization of Motion*, Computing Centre of the Russian Academy of Sciences, Moscow, 1995 (in Russian).

[7] G. Capriz, *Continua with Microstructure*, Springer Tracts in Natural Philosophy, vol. **35**, Springer-Verlag, New York-Berlin-Heidelberg-Paris-Tokyo, 1989.

[8] G. Capriz and P. M. Mariano, *Symmetries and Hamiltonian Formalism for Complex Materials*, Journal of Elasticity **72** (2003), 57–70.

[9] D. P. Chevallier, *On the Foundations of Ordinary and Generalized Rigid Body Dynamics and the Principle of Objectivity*, Arch. Mech. **56** (2004), no. 4, 313–353.

[10] A. C. Eringen, *Nonlinear Theory of Continuous Media*, McGraw-Hill Book Company, New York, 1962.

[11] A. C. Eringen, *Mechanics of Micromorphic Continua*, in: Proceedings of the IUTAM Symposium on Mechanics of Generalized Continua, Freudenstadt and Stuttgart, 1967, E. Kröner (ed.), vol. **18**, Springer, Berlin-Heidelberg-New York, 1968, 18–33.

[12] H. Goldstein, *Classical Mechanics*, Addison-Wesley, Reading, Mass, 1950.

[13] B. Gołubowska, *Motion of Test Rigid Bodies in Riemannian Spaces*, Rep. on Math. Phys. **48** (2001), no. 1/2, 95–102.

[14] B. Gołubowska, *Models of Internal Degrees of Freedom Based on Classical Groups and Their Homogeneous Spaces*, Rep. on Math. Phys. **49** (2002), no. 2/3, 193–201.

[15] B. Gołubowska, *Affine Models of Internal Degrees of Freedom and Their Action-Angle Description*, Rep. on Math. Phys. **51** (2003), no. 2/3, 205–214.

[16] B. Gołubowska, *Infinitesimal Affinely-Rigid Bodies in Riemann Spaces*, in: Proceedings of Institute of Mathematics of NAS of Ukraine, eds: A. G. Nikitin, V. M. Boyko, R. O. Popovych, and I. A. Yehorchenko, **50**, Part 2, Institute of Mathematics, Kyiv, 2004, 774–779.

[17] B. Gołubowska, *Action-Angle Analysis of Some Geometric Models of Internal Degrees of Freedom*, J. of Nonlinear Math. Phys. **11** (2004), Supplement, 138–144.

[18] B. Gołubowska, *An Affinely-Rigid Body in Manifolds and Spaces with the Constant Curvature*, PhD thesis, Warsaw, 2006.

[19] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Interscience Publishers, New York, 1963.

[20] V. V. Kozlov and A. O. Harin, *Kepler's Problem in Constant Curvature Spaces*, Cel. Mech. Dyn. Astronomy **54** (1992), 393–399.

[21] P. M. Mariano, *Configuration Forces in Continua with Microstructure*, Z. angew. Math. Phys. **51** (2000), 752–791.

[22] J. E. Marsden and T. Ratiu, *Introduction to Mechanics and Symmetry*, Springer, New York, 1994.

[23] J. E. Marsden and T. Ratiu, *Introduction to Mechanics and Symmetry. A Basic Exposition of Classical Mechanical Systems* (second ed.), Springer, New York, 1999.

[24] A. Martens, *Dynamics of Holonomically Constrained Affinely-Rigid Body*, Rep. on Math. Phys. **49** (2002), no. 2/3, 295–303.

[25] A. Martens, *Quantization of an Affinely-Rigid Body with Constraints*, Rep. on Math. Phys. **51** (2003), no. 2/3, 287–295.

[26] A. Martens, *Hamiltonian Dynamics of Planar Affinely-Rigid Body*, J. of Nonlinear Math. Phys. **11** (2004), Supplement, 145–150.

[27] A. Martens, *Quantization of the Planar Affinely-Rigid Body*, J. of Nonlinear Math. Phys. **11** (2004), Supplement, 151–156.

[28] E. E. Rożko, *Dynamics of Affinely-Rigid Bodies with Degenerate Dimension*, Rep. on Math. Phys. **56** (2005), no. 3, 311–332.

[29] M. B. Rubin, *On the Theory of a Cosserat Point and Its Application to the Numerical Solution of Continuum Problems*, ASME J. Appl. Mech. **52** (1985), 368–372.

[30] M. B. Rubin, *Free Vibration of a Rectangular Parallelepiped Using the Theory of a Cosserat Point*, ASME J. Appl. Mech. **53** (1986), 45–50.

[31] J. J. Sławianowski, *Analytical Mechanics of Homogeneous Deformations*, Prace IPPT — IFTR Reports **8**, 1973 (in Polish).

[32] J. J. Sławianowski, *Analytical Mechanics of Finite Homogeneous Strains*, Arch. of Mech. **26** (1974), no. 4, 569–587.

[33] J. J. Sławianowski, *The Mechanics of an Affinely-Rigid Body*, Int. J. of Theor. Phys. **12** (1975), no. 4, 271–296.

[34] J. J. Sławianowski, *Newtonian Dynamics of Homogeneous Strains*, Arch. of Mech. **27** (1975), no. 1, 93–102.

[35] J. J. Sławianowski, *Bertrand Systems on* $SO(3, \mathbb{R})$ *and* $SU(2)$, Bulletin de l'Académie Polonaise des Sciences, Série des sciences physiques et astronomiques **28** (1980), no. 2, 83–94.

[36] J. J. Sławianowski, *The Mechanics of the Homogeneously-Deformable Body. Dynamical Models with High Symmetries*, Z. angew. Math. Mech. **62** (1982), 229–240.

[37] J. J. Sławianowski, *Analytical Mechanics of Deformable Bodies*, PWN — Polish Scientific Publishers, Warszawa-Poznań, 1982 (in Polish).

[38] J. J. Sławianowski, *An Affinely-Rigid Body and Hamiltonian Systems on* $GL(n, \mathbb{R})$, Rep. on Math. Phys. **26** (1988), no. 1, 73–119.

[39] J. J. Sławianowski, *Nonholonomic Variational Problems and Heuristics of Control Forces*, J. of Theor. and Applied Mech. **29** (1991), no. 3-4, 661–670.

[40] J. J. Sławianowski, *Bertrand Systems on Spaces of Constant Sectional Curvature. The Action-Angle Analysis*, Rep. on Math. Phys. **46** (2000), no. 3, 429–460.

[41] J. J. Sławianowski, *Group-Theoretic Approach to Internal and Collective Degrees of Freedom in Mechanics and Field Theory*, Technische Mechanik **22** (2002), no. 1, 8–13.

[42] J. J. Sławianowski, *Quantum and Classical Models Based on* $GL(n, \mathbb{R})$*-Symmetry*, in: Proceedings of the Second International Symposium on Quantum Theory and Symmetries, Kraków, Poland, July 18–21, 2001, eds: E. Kapuścik and A. Horzela, World Scientific, New Jersey-London-Singapore-Hong Kong, 2002, 582–588.

[43] J. J. Sławianowski, *Linear Frames in Manifolds, Riemannian Structures and Description of Internal Degrees of Freedom*, Rep. on Math. Phys. **51** (2003), no. 2/3, 345–369.

[44] J. J. Sławianowski, *Classical and Quantum Collective Dynamics of Deformable Objects. Symmetry and Integrability Problems*, in: Proceedings of the Fifth International Conference on Geometry, Integrability and Quantization, June 5–12, 2003, Varna, Bulgaria, eds: Ivaïlo M. Mladenov and Allen C. Hirshfeld, SOFTEX, Sofia, 2004, 81–108.

[45] J. J. Sławianowski, *Geodetic Systems on Linear and Affine Groups. Classics and Quantization*, J. of Nonlinear Math. Phys. **11** (2004), Supplement, 130–137.

[46] J. J. Sławianowski and V. Kovalchuk, *Invariant Geodetic Problems on the Affine Group and Related Hamiltonian Systems*, Rep. on Math. Phys. **51** (2003), no. 2/3, 371–379.

[47] J. J. Sławianowski and V. Kovalchuk, *Invariant Geodetic Problems on the Projective Group* $Pr(n, \mathbb{R})$, Proceedings of Institute of Mathematics of NAS of Ukraine, eds: A. G. Nikitin, V. M. Boyko, R. O. Popovych, and I. A. Yehorchenko, **50**, Part 2, Kyiv, Institute of Mathematics, 2004, 955–960.

[48] J. J. Sławianowski and V. Kovalchuk, *Classical and Quantized Affine Physics. A Step Towards It*, J. of Nonlinear Math. Phys. **11** (2004), Supplement, 157–166.

[49] J. J. Sławianowski, V. Kovalchuk, A. Sławianowska, B. Gołubowska, A. Martens, E. E. Rożko, and Z. J. Zawistowski, *Invariant Geodetic Systems on Lie Groups and Affine Models of Internal and Collective Degrees of Freedom*, Prace IPPT — IFTR Reports **7**, 2004.

[50] J. J. Sławianowski, V. Kovalchuk, A. Sławianowska, B. Gołubowska, A. Martens, E. E. Rożko, and Z. J. Zawistowski, *Affine Symmetry in Mechanics of Collective and Internal Modes. Part I. Classical Models*, Rep. on Math. Phys. **54** (2004), no. 3, 373–427.

[51] J. J. Sławianowski, V. Kovalchuk, A. Sławianowska, B. Gołubowska, A. Martens, E. E. Rożko, and Z. J. Zawistowski, *Affine Symmetry in Mechanics of Collective and Internal Modes. Part II. Quantum Models*, Rep. on Math. Phys. **55** (2005), no. 1, 1–45.

[52] E. Sousa Dias, *A Geometric Hamiltonian Approach to the Affine Rigid Body*, in: *Dynamics, Bifurcation and Symmetry. New Trends and New Tools*, P. Chossat (ed.), NATO ASI Series C, vol. **437**, Kluwer Academic Publishers, Netherlands, 1994, 291–299.

[53] S. Sternberg, *Lectures on Differential Geometry*, Prentice-Hall, New York, 1964.

[54] J. L. Synge, *Classical Dynamics*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960.