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GALILEAN-INVARIANT FORMULATION OF THE FLUID MECHANICS



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Preface

It seems that an approach, discussed in this text, started in 1991; then the notions of the “amorphous Galilean space-time” and the “Galilean space-time with measurable time intervals” have been introduced in Sławianowski’s monograph [1] and the notion of the “non-relativistic four-velocity” has been introduced in [2] by Piekarski. In the present text applications of the non-relativistic four-velocity in fluid mechanics and kinetic theory are discussed. The most direct application of the non-relativistic four-velocity is that it allows one to construct inertial coordinate systems explicitly, that is, in terms of the four-dimensional affine geometry (see [3]).

The Galilean space-time and the Minkowski space-time are both four-dimensional affine spaces and an important difference is that the first one possess a “canonical” family of parallel hyperplanes of simultaneous events. In order to analyse Galilean-invariant field equations (like Navier–Stokes–Fourier theory) one has to understand the nature of differential operators on the “amorphous Galilean space-time” and related spaces [1].

In 1992, the differential operators on the amorphous Galilean space-time have been introduced using the “dual” approach of Peradzyński by Piekarski (compare [4, 5, 6], see also [7, 8, 9, 10]). Alternatively, one can use the definition of the complete derivative in the normed affine space given in Schwartz’s monograph [11]. This definition can be applied in finite-dimensional affine spaces since all norms in the corresponding translation spaces are equivalent. In Galilean space-time (which is a four-dimensional affine space) the hyperplanes of simultaneous events are the three-dimensional affine spaces what implies a coexistence of two “canonical” complete derivatives; one is the “four-dimensional” and the second one is the “three-dimensional” (some results on that subject are given in [3], together with the observation that the “substantial derivative” of the fluid mechanics is a directional derivative along the non-relativistic four-velocity).

In the present text it is shown that the Navier–Stokes–Fourier equations can be written invariantly. The invariant interpretation of the Gibbs identity is given (see Eqs. (3.43)–(3.53)).

Some invariant aspects of the non-relativistic kinetic theory are also discussed.

The potential application of our approach is the problem of the symmetry group for the fluid mechanics and the kinetic theory. As it is well-known, in continuum mechanics one usually applies the “principle of the material indifference” (see, for example, Jemioło and Telega [12]) but at the same time some scientists stress that in the kinetic theory of gases such quantities as the heat flux have non-objective macroscopical constitutive laws ([13], p. 97).

It is not excluded that the above mentioned discrepancy could be eliminated after formulating the fluid mechanics and the kinetic theory in the manner invariant with respect to the automorphisms of the Galilean space-time. Our hypotheses on this subject is discussed shortly in the last chapter and in Appendix F (the adequate formalism here is Rychlewski’s theory of “ Γ -structures” [45] for “affine” automorphisms of Galilean group acting on the Galilean space-time).

Readers uninterested in Galilean invariance can read the second chapter only, where the results of this text concerning the Navier–Stokes–Fourier equations are written in the standard notation. In particular, new solutions of the Gibbs identity for dense fluids are found and the corresponding sound speeds are computed. It is hoped that our approach shall be applied in acoustics of fluids (part of our main results shall be published in Archives of Acoustics, [17]). In medical acoustics, biological tissues are often modelled as the dense fluids [47, 67] what gives strong motivation for developing of mathematical methods in the modelling of dense fluids.

Chapter 1

Introduction

It is well-known that the Galilean space-time has a structure of a four-dimensional affine space and it carries many other structures besides the affine one (see [1, 14, 15]). In particular, the definition of “the amorphous Galilean space-time” has been introduced on p. 382 of [1] and the definition of “the Galilean space-time with measurable time intervals” has been introduced on p. 383 of [1]. The meaning of these definitions has been commented by Sławianowski on p. 374 of [1]:

... “We shall always order spatial and temporal structures hierarchically from the weakest to the strongest one. For example, we will start with the amorphous affine geometry in space and time and try to develop kinematic and dynamic concept as far as possible without metric structures. Later on we show what simplifications arise if we make use of metric concepts. This approach enables us to understand properly the mechanical consequences of any element of the geometric structure”...

In [2] the notion of the “non-relativistic four velocity” has been introduced and the most direct application of the non-relativistic four-velocity is that it allows one to construct inertial coordinate systems explicitly. In order to show that, one has to start with the definition of the Galilean space-time [1, 2, 3]; it is the four-dimensional affine space G with “additional structures”:

$$G = (G', T_G, -, \Psi, \cdot), \quad (1.1)$$

where G' is a set of points of the affine space, T_G is its translation space, “ $-$ ” is an operation of subtracting points of the affine space, Ψ is called the chronological form (that is, a fixed non-zero form from T_G^* , where T_G^* denotes the vector space dual to T_G). The vector space of “spatial” vectors S is defined in the following way

$$S = \{z \in T_G; \langle \Psi, z \rangle = 0\}, \quad (1.2)$$

and “ \cdot ” denotes the scalar product in S . The alternative notation for this scalar product is “ $(\ , \)$ ”.

In (1.1) a separate notation has been introduced for the “Galilean space-time as the algebraic structure” (as G) and for the “Galilean space-time as the set of space-time points on which acts the Galilean group” (as G'). For simplicity, in the rest of this text we shall use “ G ” in both cases.

The physical meaning of the vector space S is that the spaces of simultaneous events are hyperplanes in G parallel to S .

In general, in order to discuss the invariant relations between the Navier–Stokes–Fourier equations and “thermodynamics” one has to understand the structure of differential operators on the “amorphous Galilean space-time” and related spaces [1]. We shall use the definitions of affine space used in literature [1, 11, 20] (see also Appendix A).

In order to show that the invariant approach is equivalent to the standard one, one has to introduce “inertial coordinate systems” and in order to do that it is necessary to introduce the notion of the non-relativistic four-velocity and to describe some its properties.

The set of non-relativistic four-velocities is defined as [1, 2]

$$W = \{ \vec{z} \in T_A; \langle \Psi, \vec{z} \rangle = 1 \}, \quad (1.3)$$

where Ψ is the chronological form, standing in the definition (1.1). In order to define inertial coordinate systems we shall need two observations:

Observation 1.

If \vec{w} and \vec{w}' belong to W , then $\vec{w} - \vec{w}'$ belongs to S . In order to see that, it is sufficient to observe that $\langle \Psi, \vec{w} \rangle = 1$ and $\langle \Psi, \vec{w}' \rangle = 1$ implies that $\langle \Psi, \vec{w} \rangle - \langle \Psi, \vec{w}' \rangle = 0$, further $\langle \Psi, \vec{w} - \vec{w}' \rangle = 0$ and finally $\vec{w} - \vec{w}' \in S$.

Observation 2.

The sum of an arbitrary element of W and of the arbitrary element of S belongs to do W .

It follows directly from the corresponding definitions: if $\vec{w} \in W$ and $\vec{s} \in S$ then $\langle \Psi, \vec{w} + \vec{s} \rangle = \langle \Psi, \vec{w} \rangle + \langle \Psi, \vec{s} \rangle = 1 + 0 = 1$, therefore $\vec{w} + \vec{s} \in W$.

In non-relativistic physics, one uses frequently a notion of an inertial observer and it is possible to define it in terms of four-dimensional affine geometry; namely the world-line of an inertial observer can be defined parametrically as

$$R \ni t \rightarrow g + t\vec{w} \in G. \quad (1.4)$$

Such inertial observer is determined by a pair (g, \vec{w}) belonging to the product $G \times W$ (such identification is not unique but it is not important here). The

inertial coordinate system, describing observations of the inertial observer with the world-line (1.4) can be defined as

$$R^4 \ni (t, x^\alpha) \rightarrow g + t\vec{w} + x^\alpha \vec{e}_\alpha \in G. \quad (1.5)$$

In (1.5), \vec{e}_α , $\alpha = 1, 2, 3$ form an orthonormal basis in S . In order to show that properties of the coordinate systems of the form (1.5) are consistent with standard rules for Galilean transformations, one has to consider two such coordinate systems and to discuss transformation rules between them. Let us introduce another inertial observer, determined by a pair (g', \vec{w}') and with a world-line defined parameterically as

$$R \ni t \rightarrow g' + t\vec{w}' \in G.$$

The coordinates of the another inertial coordinate system shall be t' and x'^α and the explicit form of “primed” coordinate system is

$$R^4 \ni (t', x'^\alpha) \rightarrow g' + t'\vec{w}' + x'^\alpha \vec{e}_\alpha \in G. \quad (1.6)$$

If a given point g'' of the Galilean space-time G is given in both coordinate systems (1.5) and (1.6) simultaneously, then the following identity holds:

$$g'' = g + t\vec{w} + x^i \vec{e}_i = g' + t'\vec{w}' + x'^i \vec{e}_i. \quad (1.7)$$

According to the rules given in [11, 15, 19] (see also Appendix A), the relation (1.7) can be equivalently written as

$$\overrightarrow{g - g'} + t\vec{w} + x^i \vec{e}_i = t'\vec{w}' + x'^i \vec{e}_i. \quad (1.8)$$

The Galilean space-time is the four-dimensional affine space and therefore a “difference” of its points is the vector from the corresponding translation space. Consequently, the “difference” of g and g' is a vector from T_G

$$\overrightarrow{g - g'} \in T_G. \quad (1.9)$$

Now, T_G is a vector space and therefore it can be decomposed in any basis. However, in order to determine the explicit form of Galilean transformation from affine geometry it is convenient to use bases in T_G that are composed of an arbitrary vector from W (see (1.3)) and from a basis in S . A particular form of such a basis is given as $\{\vec{w}', \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and therefore the difference $\overrightarrow{g - g'}$ can be decomposed in such a basis in the following way:

$$\overrightarrow{g - g'} = \tau \vec{w}' + \gamma^i \vec{e}_i. \quad (1.10)$$

We shall also need the explicit expression for $\vec{w} - \vec{w}'$ and we can obtain it from the **Observation 1** from p. 2, which states that the difference $\vec{w} - \vec{w}'$ belongs to S and therefore can be decomposed in the basis \vec{e}_α , $\alpha = 1, 2, 3$

$$\vec{w} - \vec{w}' = \beta^i \vec{e}_i. \quad (1.11)$$

From (1.11) one can see that $\vec{w} = \vec{w}' + \beta^i \vec{e}_i$ can be inserted into (1.8) in place of \vec{w} . From (1.10) one can see that $\vec{g} - \vec{g}'$ can be also expressed in our basis and therefore (1.8) becomes

$$[\tau \vec{w}' + \gamma^i \vec{e}_i] + t [\vec{w}' + \beta^i \vec{e}_i] + x^i \vec{e}_i = t' \vec{w}' + x'^i \vec{e}_i. \quad (1.12)$$

In turn, (1.12) implies

$$[\tau + t - t'] \vec{w}' + [\gamma^i + t\beta^i + x^i - x'^i] \vec{e}_i = 0 \quad (1.13)$$

and finally one obtains the following identity

$$\tau + t - t' = 0 \quad (1.14)$$

and three identities

$$\gamma^i + t\beta^i + x^i - x'^i = 0 \quad (1.15)$$

for $i = 1, 2, 3$. It can be easily checked that (1.14) and (1.15) describe the “standard” Galilean transformation.

The set of all coordinate systems of the form (1.5) is called an inertial atlas [3]. As we shall see later, the tensor fields on the Galilean space-time can be conveniently described in that atlas. Such a description is analogous to the description of the tensor fields on the Minkowski space in the “Lorentz frames”. Both cases are particular cases of the description of the tensor fields on the four-dimensional affine spaces, described in the affine coordinate systems. This observation is obvious since the Galilean space-time and the Minkowski space-time are both the four-dimensional affine spaces. The geometry of affine spaces is described in detail in the L. Schwartz’s monograph [11], where the general definition of the complete derivative of the mapping between the normed affine spaces has been given. That definition can be applied also for the finite dimensional affine spaces since all norms on the finite dimensional vector spaces are equivalent. In [3] it has been observed that the “substantial derivative” of the fluid mechanics is a directional derivative along the non-relativistic four-velocity (by the non-relativistic four-velocity one means vector fields on G taking the values in W [2]).

The absolute time of the point $g \in G$ with respect to the point $g' \in G$ is defined as

$$\langle \Psi, \overrightarrow{g - g'} \rangle \quad (1.16)$$

while the “time distance” between the points g and g' is given as

$$\text{dist}(g, g') = \left| \langle \Psi, \overrightarrow{g - g'} \rangle \right|, \quad (1.17)$$

where $\left| \langle \Psi, \overrightarrow{g - g'} \rangle \right|$ denotes the modulus of $\langle \Psi, \overrightarrow{g - g'} \rangle$ [1, 2]. The equivalence relation, stating that the points g and g' are simultaneous is

$$g \sim g' \quad \text{iff} \quad \overrightarrow{g - g'} \in S \quad (1.18)$$

and the equivalence class of the point $g \in G$ can be denoted as $[g]$. Sometimes, in order to stress the fact that such an equivalence class has a structure of a three-dimensional Euclidean point space, we shall denote it $H_{[g]}$ (in mechanics, by the “Euclidean point space” one usually means an affine space with the scalar product in the translation space, see [1, 11, 20] and Appendix A). The set of all equivalence classes of the relation (1.18) can be denoted $[G]$. The mapping $g \rightarrow [g]$ can be interpreted in terms of the world-lines of the inertial observers. Let us consider parametrical descriptions of the inertial observers of the form (1.4), that is

$$R \ni t \rightarrow g + t\vec{w} \in G \quad (1.19)$$

and let us consider sets of points of G equivalent to the points of a fixed world-line. We shall use the notation “ $[g] + t$ ” in order to define the set of points equivalent to $g + t\vec{w}$ and it can be easily checked that that definition does not depend on the choice of the corresponding representatives. Both notations can be combined in the symbol $H_{[g]+t}$.

It can be easily checked that $[G]$ has a structure of a one-dimensional affine space and that the mappings

$$R \ni t \rightarrow [g] + t \in [G]$$

form an affine atlas on $[G]$.

The field of non-relativistic four-velocity shall be denoted as $\overrightarrow{c(g)}$ (see [3]):

$$G \ni g \rightarrow \overrightarrow{c(g)} \in W. \quad (1.20)$$

It is worth to observe that the set of the non-relativistic four-velocities W has a structure of a three-dimensional Euclidean point space. That fact is important

for many applications (including the invariant form of the moment equations in the non-relativistic kinetic theory) and therefore we shall discuss it in more detail here. It follows from the **Observations 1, 2** from p. 2 and from the existence of the scalar product (\cdot, \cdot) on the space of spatial vectors in the definition of the Galilean space-time. We shall write that algebraic structure as $(W, S, -, \cdot)$. In general, in order to define affine coordinate systems on $(W, S, -, \cdot)$ one can take the arbitrary basis $\vec{E}_1, \vec{E}_2, \vec{E}_3$ in S and define the affine coordinate systems as

$$R^3 \ni (Z^1, Z^2, Z^3) \rightarrow \vec{w} + Z^\alpha \vec{E}_\alpha \in W, \quad (1.21)$$

where $\vec{w} \in W$. However, here it is useful to take the orthonormal bases in S only and the corresponding atlas on $(W, S, -, \cdot)$ is defined as the set of mappings of the form

$$R^3 \ni (u^1, u^2, u^3) \rightarrow \vec{w} + u^\alpha \vec{e}_\alpha \in W, \quad (1.22)$$

where $\vec{w} \in W$ and $(\vec{e}_\alpha, \vec{e}_\beta) = \delta_{\alpha\beta}$ with $\alpha, \beta = 1, 2, 3$ and $\delta_{\alpha\beta}$ being the Kronecker's symbol. Therefore, any field of the non-relativistic four-velocity $\overrightarrow{c}(g)$ can be written in the form:

$$G \ni g \rightarrow \overrightarrow{c}(g) = \vec{w} + u^\alpha(g) \vec{e}_\alpha \in W. \quad (1.23)$$

Obviously, the coordinate functions $u^\alpha(g)$ depend on the choice of \vec{w} and in order to take that dependence explicitly into account in our notation we shall sometimes write

$$\overrightarrow{c}(g) = \vec{w} + u_w(g)^\alpha \vec{e}_\alpha = \vec{w} + \overrightarrow{c_w}(g). \quad (1.24)$$

In this text, invariant formulation of fluid mechanics is applied to Navier–Stokes–Fourier equations and to the Boltzmann kinetic theory.

The results concerning Navier–Stokes–Fourier equation are described in the second chapter using the standard notation, for the convenience of the readers uninterested in Galilean invariance. In general, for ideal gases the choice of the primitive fields is not important since all possible choices are equivalent. However, for fluids with the properties radically different than the ideal gases different choices are not equivalent and in the present text the density ρ and the temperature T are the primitive fields (compare [16, 17]). In order to write the Navier–Stokes–Fourier equations explicitly, one has to define such functions like, for example, the energy density $E(\rho, T)$ (per unit mass) as a function of the mass density ρ and the temperature T and the pressure $p(\rho, T)$ as a function of ρ and T . The energy density $E(\rho, T)$ as a function of ρ and T gives an important information on the fluid. According to [16, 17], the case $E(T)$ can be called “a generalized ideal gas”. In turn, by “a dense fluid” we mean medium with the energy density (per unit mass) depending not only on the temperature but also

on the mass density. The restrictions on $E(\rho, T)$ and $p(\rho, T)$ imposed by the existence of entropy are solved in the second chapter. Simple models of the dense fluids with the corresponding sound speeds are discussed shortly. Our approach is compared with “virial coefficients” (compare [17, 42]).

Motivation for developing of mathematical models of dense fluids is given, among others, by medical acoustics where biological tissues are often modelled as the dense fluids [47, 67]. In acoustics (in particular, in medical acoustics) one often applies linearizations of the full Navier–Stokes–Fourier system or “approximations” with diminished number of primitive fields (like, for example, Westervelt equation, Burgers equation or Kuznetsov equation) [47, 67, 68]. The choice of a particular approximation depends on the values of the transport coefficients and on the kind of the initial and boundary value problem; it is not excluded that the formulation of the dense fluids discussed in [17] and in the present text could be of some help here.

The invariant notation for Navier–Stokes–Fourier equations (including the invariant definition of the substantial derivative) is discussed in the third Chapter. The invariant interpretation of the Gibbs identity is given (see Eqs. (3.43)–(3.53)). The differential operators necessary for that are discussed in Appendices.

The important part of non-relativistic physics is related to different aspects of the Boltzmann kinetic equation. In principle, Boltzmann equation is a single-particle Liouville equation with a source term and therefore in order to write invariantly Boltzmann equation one has to give an invariant definition of such Liouville equation first. A rigorous definition of the invariant form of the single-particle Liouville equation requires some discussion and its elements are relegated to Appendix E.

Elementary expressions from the kinetic theory (including locally equilibrium Maxwell distribution functions) are invariantly written in Chapter four.

It is well-known that moment identities for Boltzmann equation are derived from the Boltzmann equation. A general form of such moment equations (in a standard notation) has been derived in 1989 by Banach and Piekarski [56]. In order to derive the “invariant” form of moment identities, one needs a corresponding form of the Boltzmann equation and it is given in Chapter five.

In order to give some introductory remarks on the kinetic theory, a part of p. 56 of Huang’s textbook [19] is given:

...“We are not interested in the motion of each molecule in detail. Rather, we are interested in the distribution function $f(\vec{r}, \vec{v}, t)$ so defined that

$$f(\vec{r}, \vec{v}, t) d^3\vec{r} d^3\vec{v}$$

is the number of molecules which, at time t , have positions lying within a volume element $d^3\vec{r}$ about \vec{r} and velocities lying within a velocity-space element $d^3\vec{v}$

about \vec{v} . The volume elements $d^3\vec{r}$ and $d^3\vec{v}$ are not to be taken literally as mathematically infinitesimal quantities. They are finite volume elements which are large enough to contain a very large number of molecules and yet small enough so that compared to macroscopic dimensions they are essentially points. That such a choice is possible can be seen by an example. Under standard conditions there are about 3×10^{19} molecules/cc in a gas. If we choose $d^3\vec{r} \sim 10^{-10}$ cc, which to us is small enough to be called a point, there are still on the order of 3×10^9 molecules in $d^3\vec{r}$."

It can be observed that the "invariant" counterpart of the distribution function $f(\vec{r}, \vec{v}, t)$ is a non-negative function on the product $G \times W$. In turn, the invariant counterpart of the " μ -space" (see p. 56 of [19]) is a product $H_{[g]} \times W$.

In order to show it explicitly one can start with text from p. 57 of [19] where Huang writes:

... "Having defined the distribution function, we can express the information that there are N molecules in the volume V through the normalization condition

$$\int f(\vec{r}, \vec{v}, t) d^3\vec{r} d^3\vec{v} = N.$$

If the molecules are uniformly distributed in space, so that f is independent of \vec{r} , then

$$\int f(\vec{r}, \vec{v}, t) d^3\vec{v} = \frac{N}{V}.$$

The aim of kinetic theory is to find the distribution function $f(\vec{r}, \vec{v}, t)$ for a given form of molecular interaction."

In order to compare Huang's expressions with a non-negative function on $H_{[g]} \times W$

$$H_{[g]} \times W \ni (g, \vec{w}) \rightarrow f(g, \vec{w}) \in R_+ \cup \{0\} \quad (1.25)$$

let us integrate (1.25) with respect to the variable \vec{w} from W , with g being a parameter. We already know that W is an Euclidean point space and therefore it possesses an Euclidean volume measure.

In order to conform ourselves to the conventions of the kinetic theory we have to introduce the notion of the "invariant molecular velocity" (which means a variable \vec{w} from W , standing as one of arguments of a distribution function) and "standard molecular velocity" \vec{u} (which is a molecular velocity observed by a given inertial observer). The relation between them is

$$W \ni \vec{w} = \vec{w}_I + \vec{u}, \vec{u} \in S. \quad (1.26)$$

The corresponding transformation of the "volume element" is

$$d^3\vec{w} = d^3\vec{u} \quad (1.27)$$

and after making use of the identity

$$\vec{u} = \vec{w} - \vec{w}_I \quad (1.28)$$

one can compute the integral

$$\int_W f(g, \vec{w}) d^3 \vec{w} = \int_S f(g, \vec{u}) d^3 \vec{u}. \quad (1.29)$$

From these expressions it can be seen that the integrals in the kinetic theory can be equivalently computed either “invariantly” or “in a fixed inertial coordinate systems”. For a given space-time point g_0 , for an arbitrary four-velocity $\vec{w}_I \in W$ and for arbitrary orthonormal basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$ in S , one can introduce the following inertial coordinate system on the product $G \times W$:

$$R^7 \ni (t, x^1, x^2, x^3, u^1, u^2, u^3) \rightarrow (g_0 + t\vec{w}_I + x^\alpha \vec{e}_\alpha, u^\beta \vec{e}_\beta) \in G \times W. \quad (1.30)$$

Every real function $f(g, \vec{w})$ on the product $G \times W$ can be written in the coordinates (1.30)

$$f(g, \vec{w}) = f(g_0 + t\vec{w}_I + x^\alpha \vec{e}_\alpha, u^\beta \vec{e}_\beta) \quad (1.31)$$

and sometimes we shall use the notation

$$f(g, \vec{w}) = f_{g_I, \vec{w}_I}(t, x^\alpha, u^\beta). \quad (1.32)$$

The another potential application of our approach concerns the problem of the symmetry group of the fluid mechanics and the kinetic theory. The well-known discrepancy between the “principle of material objectivity” and the non-objective effects from the kinetic theory of gases is described, for example, in [13] on p. 97. It is not excluded that this discrepancy could be eliminated after formulating the symmetry groups of continuum mechanics and the non-relativistic kinetic theory in terms of the “affine” automorphisms of the Galilean space-time and the “ μ -space”. This aspect is discussed in the Chapter six and in Appendices; in particular, our hypothesis on the alternative derivation of the constitutive restrictions (usually derived by means of the “principle of material objectivity”) is shortly described in Appendix F. From the mathematical point of view, the general scheme applied in this text is that of Sławianowski [1]; one starts with different “models” of the Galilean space-time described in different Appendices and among their “affine” automorphisms one search for such a class of automorphisms that potentially could be an alternative to “principle of material objectivity”. In general, for all such “models” it is possible to introduce the corresponding “ Γ -structures” which are related to Γ -structures on the normed affine spaces (see

remarks on the additional structures imposed on the set-theoretical background of the transformation groups given on p. 70 of Rychlewski's monograph [45]). However, the present text is of a preliminary character and a detailed discussion of these aspects is outside its scope.

Some potential "physical" generalizations of our approach are mentioned in Chapter six. Moreover, the "nonrelativistic" invariants discussed in the present text are shortly compared with the invariants for gauge-invariant formulation of perturbation calculus in general relativity, introduced by Banach and Piekarski in 1994-2000 (see [58–65]).

Chapter 2

The Navier–Stokes–Fourier equations and thermostatics. Standard notation

In this chapter the relations of the Navier–Stokes–Fourier equations and thermostatics are discussed in the standard notation. Some results are already published or accepted for publication ([16, 17]) but our present discussion is much more general. It is commonly accepted that the existence of entropy imposes restrictions on the constitutive functions in the Navier–Stokes–Fourier equations and in [16] it has been shown that if the energy per unit mass is a function of the temperature T only, then the pressure p is the arbitrary function of the density ρ multiplied by the temperature T . Now the general form of the relations between the energy density and the pressure is given (both quantities are understood as functions of the mass density and the temperature). We shall see that these relations can be approximated in a different ways and the different approximations suggest the different classifications of dense fluids (some of them are similar to the virial expansions).

For completeness, we shall give a short discussion of the Navier–Stokes–Fourier equations, beginning with the symmetric Cauchy stress tensor.

The symmetric Cauchy stress tensor can be written as the sum of the two terms; the first term is the product of the pressure p and the identity tensor (taken with the minus sign) and the second is the “dissipative part of the stress tensor”:

$$t_{ij} = -p(\rho, T)\delta_{ij} + t'_{ij}. \quad (2.1)$$

In the above formula δ_{ij} denotes the identity tensor, t_{ij} and t'_{ij} are the symmetric Euclidean tensors of the rank two, ρ denotes the mass density and T denotes the temperature.

The mass density balance has a form

$$\frac{D}{Dt}\rho + \rho \frac{\partial u^i}{\partial x^i} = 0 \quad (2.2)$$

where u^i denotes the velocity field and $\frac{D}{Dt}$ is the substantial derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u^i \frac{\partial}{\partial x^i} \quad (2.3)$$

with $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x^i}$ being the time derivative and the “spatial” derivative, correspondingly.

The momentum balance is

$$\rho \frac{D}{Dt}u^i = \frac{\partial}{\partial x^j}t_{ij} = \frac{\partial}{\partial x^i}[-p(\rho, T)\delta_{ij} + t'_{ij}], \quad (2.4)$$

and the energy balance reads

$$\rho \frac{D}{Dt}E = t_{ij} \frac{\partial u^i}{\partial x^j} - \frac{\partial q^i}{\partial x^i}, \quad (2.5)$$

where q^i is the heat flux. The explicit form of the “dissipative part of the stress tensor” for the Navier-Stokes equations is [44]

$$t'_{ij} = -\mu \frac{2}{3} \delta_{ij} \frac{\partial u^k}{\partial x^k} + \mu \left[\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} \right], \quad (2.6)$$

where μ is the viscosity coefficient.

After inserting (2.1) into (2.5) one obtains

$$\rho \frac{D}{Dt}E = [-p(\rho, T)\delta_{ij} + t'_{ij}] \frac{\partial u^i}{\partial x^j} - \frac{\partial q^i}{\partial x^i}. \quad (2.7)$$

The above formula can be written equivalently as

$$\rho \frac{D}{Dt}E = -p(\rho, T) \frac{\partial u^k}{\partial x^k} + t'_{ij} \frac{\partial u^i}{\partial x^j} - \frac{\partial q^i}{\partial x^i}. \quad (2.8)$$

In order to eliminate $\frac{\partial u^k}{\partial x^k}$ from (2.8), one can determine $\frac{\partial u^k}{\partial x^k}$ from (2.2):

$$\frac{\partial u^k}{\partial x^k} = -\frac{1}{\rho} \frac{D}{Dt}\rho, \quad (2.9)$$

and insert (2.9) into (2.8), what gives

$$\begin{aligned} \rho \frac{D}{Dt} E &= -p(\rho, T) \left[-\frac{1}{\rho} \frac{D}{Dt} \rho \right] + t'_{ij} \frac{\partial u^i}{\partial x^j} - \frac{\partial q^i}{\partial x^i} = \\ &= \frac{p(\rho, T)}{\rho} \frac{D}{Dt} \rho + t'_{ij} \frac{\partial u^i}{\partial x^j} - \frac{\partial q^i}{\partial x^i}. \end{aligned} \quad (2.10)$$

Now, (2.10) is equivalent to

$$\rho \left\{ \frac{D}{Dt} E - \frac{p(\rho, T)}{\rho^2} \frac{D}{Dt} \rho \right\} = t'_{ij} \frac{\partial u^i}{\partial x^j} - \frac{\partial q^i}{\partial x^i}. \quad (2.11)$$

After introducing

$$\frac{D}{Dt} \frac{1}{\rho} = -\frac{1}{\rho^2} \frac{D}{Dt} \rho \quad (2.12)$$

into (2.11) one obtains

$$\rho \left\{ \frac{D}{Dt} E + p(\rho, T) \frac{D}{Dt} \frac{1}{\rho} \right\} = t'_{ij} \frac{\partial u^i}{\partial x^j} - \frac{\partial q^i}{\partial x^i}. \quad (2.13)$$

For the temperatures different from zero, the above equation can be multiplied by the inverse of the temperature

$$\rho \left\{ T^{-1} \frac{D}{Dt} E + T^{-1} p(\rho, T) \frac{D}{Dt} \frac{1}{\rho} \right\} = T^{-1} \left\{ t'_{ij} \frac{\partial u^i}{\partial x^j} - \frac{\partial q^i}{\partial x^i} \right\}. \quad (2.14)$$

In our case the primitive fields are ρ and T and therefore the internal energy per unit mass and the pressure are of a general form

$$E = E(\rho, T), \quad (2.15)$$

$$p = p(\rho, T). \quad (2.16)$$

The Gibbs identity states that the entropy $S(\rho, T)$ satisfies the following relation (compare [13, 18])

$$\frac{D}{Dt} S(\rho, T) = T^{-1} \frac{D}{Dt} E + T^{-1} p(\rho, T) \frac{D}{Dt} \frac{1}{\rho}. \quad (2.17)$$

It can be checked that the restriction, imposed on the energy density (2.15) per unit mass and on the pressure (2.16) by the existence of the entropy $S(\rho, T)$, is

$$\frac{\partial}{\partial T} \left[T^{-1} \frac{\partial E(\rho, T)}{\partial \rho} - \frac{p(\rho, T)}{T \rho^2} \right] = \frac{\partial}{\partial \rho} \left[T^{-1} \frac{\partial E(\rho, T)}{\partial T} \right]. \quad (2.18)$$

In [16], this relation has been obtained directly from the Navier–Stokes–Fourier equations; the alternative approach is to obtain it from the Gibbs identity. The relations of the Gibbs identity and the Navier–Stokes–Fourier equations can be investigated also by means of the Lagrange’a-Liu multipliers (see detailed calculations of Wilmański in [41]). In order to discuss (2.18), let us write it in the form

$$\frac{\partial}{\partial T} \left[T^{-1} \frac{\partial E(\rho, T)}{\partial \rho} \right] + \frac{\partial}{\partial T} \left[-\frac{p(\rho, T)}{T\rho^2} \right] = T^{-1} \frac{\partial^2 E(\rho, T)}{\partial \rho \partial T}, \quad (2.19)$$

which is equivalent to

$$T^{-1} \frac{\partial^2 E(\rho, T)}{\partial T \partial \rho} + \frac{\partial E(\rho, T)}{\partial \rho} \frac{\partial}{\partial T} [T^{-1}] + \frac{\partial}{\partial T} \left[-\frac{p(\rho, T)}{T\rho^2} \right] = T^{-1} \frac{\partial^2 E(\rho, T)}{\partial \rho \partial T}, \quad (2.20)$$

It can be seen that the first term on the l.h.s. of (2.20) cancels with the term on the r.h.s. of (2.20). Therefore one arrives at

$$\frac{\partial E(\rho, T)}{\partial \rho} \frac{\partial}{\partial T} [T^{-1}] + \frac{\partial}{\partial T} \left[-\frac{p(\rho, T)}{T\rho^2} \right] = 0, \quad (2.21)$$

that is,

$$-\frac{1}{T^2} \frac{\partial E(\rho, T)}{\partial \rho} + \frac{\partial}{\partial T} \left[-\frac{p(\rho, T)}{T\rho^2} \right] = 0. \quad (2.22)$$

After multiplying (2.22) by the temperature T one obtains

$$-\frac{1}{T} \frac{\partial E(\rho, T)}{\partial \rho} + T \frac{\partial}{\partial T} \left[-\frac{p(\rho, T)}{T\rho^2} \right] = 0. \quad (2.23)$$

After multiplying (2.23) by the mass density ρ the result can be written in the following form

$$\rho \frac{\partial}{\partial \rho} \left[\frac{E(\rho, T)}{T} \right] + T \frac{\partial}{\partial T} \left[\frac{p(\rho, T)}{T\rho} \right] = 0. \quad (2.24)$$

In order to get a deeper insight into the nature of that equation, let us define the following symbols

$$\widehat{E}(\rho, T) = \frac{E(\rho, T)}{T}, \quad (2.25)$$

$$\widehat{p}(\rho, T) = \frac{p(\rho, T)}{T\rho}. \quad (2.26)$$

After taking into account (2.25) and (2.26) the equation (2.24) can be written in the form

$$\rho \frac{\partial}{\partial \rho} \widehat{E}(\rho, T) + T \frac{\partial}{\partial T} \widehat{p}(\rho, T) = 0. \quad (2.27)$$

It can be seen that a particular solution of (2.27) is

$$\widehat{E}(\rho, T) = \widehat{E}(T), \quad (2.28)$$

$$\widehat{p}(\rho, T) = \widehat{p}(\rho). \quad (2.29)$$

In order to interpret (2.28), (2.29) in “standard variables”, let us invert (2.25) and (2.26)

$$E(\rho, T) = \widehat{E}(\rho, T)T, \quad (2.30)$$

$$p(\rho, T) = \widehat{p}(\rho, T)T\rho, \quad (2.31)$$

and after inserting (2.28) and (2.29) into (2.30) and (2.31) we arrive at:

$$E(\rho, T) = \widehat{E}(T)T, \quad (2.32)$$

$$p(\rho, T) = \widehat{p}(\rho)T\rho. \quad (2.33)$$

That solution has been obtained in [16] in a different way. Its important property is that the energy density per unit mass does not depend on the mass density and that the expression for the pressure is a product of the arbitrary function of the mass density and the linear function of the temperature. That solution describes some generalization of the ideal gas and therefore the corresponding medium shall be called “a generalized ideal gas”. In turn, by a “dense fluid” we shall mean medium with the energy density (per unit mass) depending not only on the temperature but also on the mass density.

In general, if $\widehat{E}(\rho, T)$ and $\widehat{p}(\rho, T)$ satisfies (2.27), then

$$\widehat{E}'(\rho, T) = \widehat{E}(\rho, T) + \varphi(T), \quad (2.34)$$

$$\widehat{p}'(\rho, T) = \widehat{p}(\rho, T) + \gamma(\rho) \quad (2.35)$$

is the other solution of (2.27). That property can be checked easily:

$$\begin{aligned} & \rho \frac{\partial}{\partial \rho} \widehat{E}'(\rho, T) + T \frac{\partial}{\partial T} \widehat{p}'(\rho, T) = \\ & \rho \frac{\partial}{\partial \rho} [\widehat{E}(\rho, T) + \varphi(T)] + T \frac{\partial}{\partial T} [\widehat{p}(\rho, T) + \gamma(\rho)] = \end{aligned}$$

$$\begin{aligned} \rho \frac{\partial}{\partial \rho} \widehat{E}(\rho, T) + \rho \frac{\partial}{\partial \rho} \varphi(T) + T \frac{\partial}{\partial T} \widehat{p}(\rho, T) + T \frac{\partial}{\partial T} \gamma(\rho) = \\ \rho \frac{\partial}{\partial \rho} \widehat{E}(\rho, T) + T \frac{\partial}{\partial T} \widehat{p}(\rho, T). \end{aligned} \quad (2.36)$$

The above property means that the solutions of (2.27) can be divided into the equivalence classes; any two solutions are equivalent if and only if their difference is a solution for the “generalized ideal gas”. Since (2.27) is linear, the set of its solutions forms a vector space, and the above property means the existence of the corresponding quotient space in the space of solutions of (2.27).

In order to get some explicit knowledge about the solutions of (2.27) let us investigate its consequences, namely

$$\frac{\partial}{\partial \rho} \widehat{E}(\rho, T) = -\frac{T}{\rho} \frac{\partial}{\partial T} \widehat{p}(\rho, T) \quad (2.37)$$

and

$$\frac{\partial}{\partial T} \widehat{p}(\rho, T) = -\frac{\rho}{T} \frac{\partial}{\partial \rho} \widehat{E}(\rho, T) \quad (2.38)$$

(of course, we assume that both primitive fields are different from zero).

After integrating (2.37) and (2.38) with respect to the mass density and the temperature, correspondingly, one obtains

$$\int_{\rho_0}^{\rho} \frac{\partial}{\partial \rho'} \widehat{E}(\rho', T) d\rho' = \widehat{E}(\rho, T) - \widehat{E}(\rho_0, T) = -T \frac{\partial}{\partial T} \int_{\rho_0}^{\rho} \frac{\widehat{p}(\rho', T)}{\rho'} d\rho' \quad (2.39)$$

and

$$\int_{T_0}^T \frac{\partial}{\partial T'} \widehat{p}(\rho, T') dT' = \widehat{p}(\rho, T) - \widehat{p}(\rho, T_0) = -\rho \frac{\partial}{\partial \rho} \int_{T_0}^T \frac{\widehat{E}(\rho, T')}{T'} dT'. \quad (2.40)$$

Now, (2.39) implies that

$$\widehat{E}(\rho, T) = \widehat{E}(\rho_0, T) - T \frac{\partial}{\partial T} \int_{\rho_0}^{\rho} \frac{\widehat{p}(\rho', T)}{\rho'} d\rho' \quad (2.41)$$

and (2.40) implies that

$$\widehat{p}(\rho, T) = \widehat{p}(\rho, T_0) - \rho \frac{\partial}{\partial \rho} \int_{T_0}^T \frac{\widehat{E}(\rho, T')}{T'} dT'. \quad (2.42)$$

From (2.41) and (2.42) one can see the general form of the solutions of (2.27); if $\widehat{p}(\rho, T)$ is arbitrary, then $\widehat{E}(\rho, T)$ must be of the form

$$\widehat{E}(\rho, T) = \widetilde{E}(T) - T \frac{\partial}{\partial T} \Pi_\rho \left(\frac{\widehat{p}(\rho, T)}{\rho} \right), \quad (2.43)$$

where $\widetilde{E}(T)$ is the arbitrary function of the temperature T and Π_ρ is the indefinite integral with respect to the variable ρ (with the variable T being a parameter).

In turn, if $\widehat{E}(\rho, T)$ is arbitrary then $\widehat{p}(\rho, T)$ must be of the form

$$\widehat{p}(\rho, T) = \widetilde{p}(\rho) - \rho \frac{\partial}{\partial \rho} \Pi_T \left(\frac{\widehat{E}(\rho, T)}{T} \right), \quad (2.44)$$

where $\widetilde{p}(\rho)$ is the arbitrary function of the mass density ρ and Π_T is the indefinite integral with respect to the variable T (with the variable ρ being a parameter).

Obviously, the symbols Π_ρ and Π_T in (2.43) and (2.44) satisfy the relations

$$\frac{\partial}{\partial \rho} \left\{ \Pi_\rho \left(\frac{\widehat{p}(\rho, T)}{\rho} \right) \right\} = \frac{\widehat{p}(\rho, T)}{\rho} \quad (2.45)$$

and

$$\frac{\partial}{\partial T} \Pi_T \left(\frac{\widehat{E}(\rho, T)}{T} \right) = \frac{\widehat{E}(\rho, T)}{T}. \quad (2.46)$$

It is worth to mention that the indefinite integrals are defined up to the additive constants but in (2.43) and (2.44) the indefinite integrals stand under the differential operators and therefore the resulting expressions are defined uniquely.

By direct inspection, one can check that (2.43) is a solution of (2.27):

$$\begin{aligned} \rho \frac{\partial}{\partial \rho} \widehat{E}(\rho, T) &= \rho \frac{\partial}{\partial \rho} \left[\widetilde{E}(T) - T \frac{\partial}{\partial T} \Pi_\rho \left(\frac{\widehat{p}(\rho, T)}{\rho} \right) \right] = \\ &= \rho \frac{\partial}{\partial \rho} [\widetilde{E}(T)] - \rho \frac{\partial}{\partial \rho} \left[T \frac{\partial}{\partial T} \Pi_\rho \left(\frac{\widehat{p}(\rho, T)}{\rho} \right) \right] = \\ &= -\rho \frac{\partial}{\partial \rho} T \frac{\partial}{\partial T} \Pi_\rho \left(\frac{\widehat{p}(\rho, T)}{\rho} \right) = -\rho T \frac{\partial}{\partial T} \frac{\partial}{\partial \rho} \Pi_\rho \left(\frac{\widehat{p}(\rho, T)}{\rho} \right) = \\ &= -\rho T \frac{\partial}{\partial T} \frac{\widehat{p}(\rho, T)}{\rho} = -T \frac{\partial}{\partial T} \widehat{p}(\rho, T). \end{aligned} \quad (2.47)$$

Similarly, one can check that (2.44) is a solution of (2.27)

$$\begin{aligned}
T \frac{\partial}{\partial T} \widehat{p}(\rho, T) &= T \frac{\partial}{\partial T} \left[\widetilde{p}(\rho) - \rho \frac{\partial}{\partial \rho} \Pi_T \left(\frac{\widehat{E}(\rho, T)}{T} \right) \right] = \\
&= T \frac{\partial}{\partial T} [\widetilde{p}(\rho)] - T \frac{\partial}{\partial T} \left[\rho \frac{\partial}{\partial \rho} \Pi_T \left(\frac{\widehat{E}(\rho, T)}{T} \right) \right] = \\
&= -T \frac{\partial}{\partial T} \left[\rho \frac{\partial}{\partial \rho} \Pi_T \left(\frac{\widehat{E}(\rho, T)}{T} \right) \right] = -T \rho \frac{\partial}{\partial \rho} \frac{\partial}{\partial T} \Pi_T \left(\frac{\widehat{E}(\rho, T)}{T} \right) = \\
&= -T \rho \frac{\partial}{\partial \rho} \frac{\widehat{E}(\rho, T)}{T} = -\rho \frac{\partial}{\partial \rho} \widehat{E}(\rho, T). \tag{2.48}
\end{aligned}$$

It is well-known that the energy, the pressure and the entropy can be expressed in terms of the free energy (see, for example, Wilmański [41]):

$$E(\rho, T) = F(\rho, T) - T \frac{\partial F(\rho, T)}{\partial T}, \tag{2.49}$$

where $F(\rho, T)$ is the free energy per unit mass,

$$S(\rho, T) = -\frac{\partial F(\rho, T)}{\partial T} \tag{2.50}$$

is the entropy per unit mass, and the pressure is

$$p(\rho, T) = \rho^2 \frac{\partial F(\rho, T)}{\partial \rho}. \tag{2.51}$$

Our solutions of (2.27) are defined in terms of the variables $\widehat{E}(\rho, T)$ and $\widehat{p}(\rho, T)$; therefore in order to check whether (2.49) and (2.51) satisfy (2.27) one has to write (2.25) and (2.26) in the form

$$\widehat{E}(\rho, T) = \frac{F(\rho, T)}{T} - \frac{\partial F(\rho, T)}{\partial T}, \tag{2.52}$$

$$\widehat{p}(\rho, T) = \frac{\rho}{T} \frac{\partial F(\rho, T)}{\partial \rho}. \tag{2.53}$$

It can be checked easily that (2.52) and (2.53) satisfy the equation (2.27); however, our aim here is to investigate explicit relations between the energy and the pressure and therefore expressions formulated in terms of the free energy are not useful for our purposes.

Our results can be approximated in many ways and some of them seem to be related to the virial coefficients [48].

Generally speaking, since the equation (2.27) is linear, one can choose different bases in the space of its solutions. In order to interpret the integral operators Π_ρ and Π_T explicitly, it is convenient to use the functions that are products of single variables. For example, one can assume that $\widehat{E}(\rho, T)$ is

$$\widehat{E}(\rho, T) = \widehat{E}_0(T) + \widehat{E}_1(T)\rho + \widehat{E}_2(T)\rho^2 + \dots + \widehat{E}_n(T)\rho^n, \quad (2.54)$$

and insert (2.54) into (2.42):

$$\begin{aligned} \widehat{p}(\rho, T) &= \\ &= \widehat{p}(\rho, T_0) - \rho \frac{\partial}{\partial \rho} \int_{T_0}^T \frac{[\widehat{E}_0(T') + \widehat{E}_1(T')\rho + \widehat{E}_2(T')\rho^2 + \dots + \widehat{E}_n(T')\rho^n]}{T'} dT' = \\ &\widehat{p}(\rho, T_0) - \rho \frac{\partial}{\partial \rho} \int_{T_0}^T \frac{\widehat{E}_1(T')\rho}{T'} dT' + \dots \rho \frac{\partial}{\partial \rho} \int_{T_0}^T \frac{\widehat{E}_n(T')\rho^n}{T'} dT' = \\ &\widehat{p}(\rho, T_0) - \rho \frac{\partial}{\partial \rho} \rho \int_{T_0}^T \frac{\widehat{E}_1(T')}{T'} dT' + \dots \rho \frac{\partial}{\partial \rho} \rho^n \int_{T_0}^T \frac{\widehat{E}_n(T')}{T'} dT' = \\ &\widehat{p}(\rho, T_0) - \rho \int_{T_0}^T \frac{\widehat{E}_1(T')}{T'} dT' + \dots + n\rho^n \int_{T_0}^T \frac{\widehat{E}_n(T')}{T'} dT'. \end{aligned} \quad (2.55)$$

In turn, one can assume that

$$\widehat{p}(\rho, T) = \widehat{p}_0(\rho) + \widehat{p}_1(\rho)T + \widehat{p}_2(\rho)T^2 + \dots + \widehat{p}_n(\rho)T^n \quad (2.56)$$

and insert (2.56) into (2.41):

$$\begin{aligned} \widehat{E}(\rho, T) &= \\ &= \widehat{E}(\rho_0, T) - T \frac{\partial}{\partial T} \int_{\rho_0}^{\rho} \frac{[\widehat{p}_0(\rho') + \widehat{p}_1(\rho')T + \widehat{p}_2(\rho')T^2 + \dots + \widehat{p}_n(\rho')T^n]}{\rho'} d\rho' = \\ &\widehat{E}(\rho_0, T) - T \frac{\partial}{\partial T} \int_{\rho_0}^{\rho} \frac{\widehat{p}_1(\rho')T}{\rho'} d\rho' - \dots - T \frac{\partial}{\partial T} \int_{\rho_0}^{\rho} \frac{\widehat{p}_n(\rho')T^n}{\rho'} d\rho' = \end{aligned}$$

$$\begin{aligned} \widehat{E}(\rho_0, T) - T \frac{\partial}{\partial T} T \int_{\rho_0}^{\rho} \frac{\widehat{p}_1(\rho')}{\rho'} d\rho' - \dots - T \frac{\partial}{\partial T} T^n \int_{\rho_0}^{\rho} \frac{\widehat{p}_n(\rho')}{\rho'} d\rho' = \\ \widehat{E}(\rho_0, T) - T \int_{\rho_0}^{\rho} \frac{\widehat{p}_1(\rho')}{\rho'} d\rho' - \dots - nT^n \int_{\rho_0}^{\rho} \frac{\widehat{p}_n(\rho')}{\rho'} d\rho'. \end{aligned} \quad (2.57)$$

Those expressions can be used for an approximate classification of the “dense fluids”. As it has been already mentioned, by a dense fluid we mean here any fluid in which the energy density per unit mass is a function not only of the temperature but also of the mass density. Our expressions seem to be consistent with the well-known “virial expansions” [48].

The equation (2.27) is symmetric and in our notation that symmetry has been taken explicitly into account. Now we shall use the standard variables in order to compare the “generalized ideal gas” and the simple examples of dense fluids. Later, the corresponding sound speeds shall be computed.

In [16], the solution of Gibbs identity of the form (2.32), (2.33) has been given, with the energy depending on the temperature T only and the pressure being a product of the arbitrary function of the mass density $p_0(\rho)$ and the temperature T

$$E = E(T), \quad p = p_0(\rho)T. \quad (2.58)$$

The corresponding entropy density is [17]:

$$S(\rho, T) = \int_{T_0}^T \frac{1}{T'} \frac{\partial E(T')}{\partial T'} dT' - \int_{\rho_0}^{\rho} \frac{p_0(\rho')}{\rho'^2} d\rho'. \quad (2.59)$$

For (2.58) and (2.59), Gibbs identity can be explicitly checked

$$\begin{aligned} TdS(\rho, T) = Td \left\{ \int_{T_0}^T \frac{1}{T'} \frac{\partial E(T')}{\partial T'} dT' - \int_{\rho_0}^{\rho} \frac{p_0(\rho')}{\rho'^2} d\rho' \right\} = \\ dE(T) + p_0(\rho)Td \left[\frac{1}{\rho} \right] = dE(T) - \frac{p_0(\rho)T}{\rho^2} d\rho. \end{aligned} \quad (2.60)$$

In particular, (2.58) can take a form

$$E = AT, \quad p = B\rho T, \quad (2.61)$$

and the corresponding expression for the entropy is

$$\begin{aligned}
 S(\rho, T) &= \int_{T_0}^T \frac{1}{T'} \frac{\partial [AT']}{\partial T'} dT' - \int_{\rho_0}^{\rho} \frac{[B\rho]}{\rho'^2} d\rho' = \\
 &= \int_{T_0}^T \frac{A}{T'} dT' - \int_{\rho_0}^{\rho} \frac{B}{\rho'} d\rho' = A \int_{T_0}^T \frac{1}{T'} dT' - B \int_{\rho_0}^{\rho} \frac{1}{\rho'} d\rho' = \\
 &= A \left[\ln T' \right]_{T_0}^T - B \left[\ln \rho' \right]_{\rho_0}^{\rho} = A \ln \frac{T}{T_0} - B \ln \frac{\rho}{\rho_0}. \quad (2.62)
 \end{aligned}$$

In order to see that (2.62) is consistent with the standard expression for the entropy of an ideal gas it is sufficient to introduce the specific volume V

$$\frac{1}{V} = \rho$$

(compare, for example, p. 223 of [13]).

If an ideal gas has only the translational degrees of freedom (that is, the rotational and vibrational degrees of freedom are absent) the additional relation takes place [19]

$$\rho E = \frac{3}{2} p,$$

and then

$$A = \frac{3}{2} B. \quad (2.63)$$

A “dense fluid” is distinguished by a property that the energy density (per unit mass) depends not only on the temperature but also on the mass density and some models of dense fluids can be obtained from the following observation. It can be observed that among solutions of the Gibbs identity there exists an equivalence relation stating that two solutions $[E(\rho, T), p(\rho, T)]$ and $[E'(\rho, T), p'(\rho, T)]$ are equivalent if and only if the following identity takes place

$$dE(\rho, T) + p(\rho, T) d \left[\frac{1}{\rho} \right] = dE'(\rho, T) + p'(\rho, T) d \left[\frac{1}{\rho} \right]. \quad (2.64)$$

It is obvious that the above equivalence relation does not change the entropy of the considered system and it can be checked that it can relate an ideal gas with a dense fluid. In order to “solve” the condition (2.64) one can write it as a “perturbation”, transforming $[E(\rho, T), p(\rho, T)]$ into $[E'(\rho, T), p'(\rho, T)]$

$$\left[E(\rho, T), p(\rho, T) \right] \rightarrow \left[E'(\rho, T), p'(\rho, T) \right] = \left[E(\rho, T) + \Delta E, p(\rho, T) + \Delta p \right] \quad (2.65)$$

and to determine ΔE and Δp . One can easily see that ΔE and Δp are related by the condition

$$d[\Delta E] + \Delta p d\left[\frac{1}{\rho}\right] = d[\Delta E] - \frac{\Delta p}{\rho^2} d\rho = 0. \quad (2.66)$$

That condition can be written in terms of the differentials of the primitive fields

$$d[\Delta E] - \frac{\Delta p}{\rho^2} d\rho = \left[\frac{\partial \Delta E}{\partial \rho} - \frac{\Delta p}{\rho^2} \right] d\rho + \frac{\partial \Delta E}{\partial T} dT = 0 \quad (2.67)$$

what implies that

$$\begin{aligned} \frac{\partial[\Delta E]}{\partial \rho} - \frac{\Delta p}{\rho^2} &= 0, \\ \frac{\partial[\Delta E]}{\partial T} &= 0. \end{aligned} \quad (2.68)$$

Therefore, both “perturbations” can depend on the mass density only and they are related by the condition

$$\Delta p(\rho) = \rho^2 \frac{\partial \Delta E(\rho)}{\partial \rho}. \quad (2.69)$$

The above described procedure can be applied, in particular, to the “generalized ideal gas” and it transforms it into the following simple model of the dense fluid

$$E(\rho, T) = E(T) + \Delta E(\rho), \quad p = p_0(\rho)T + \rho^2 \frac{\partial \Delta E(\rho)}{\partial \rho}, \quad (2.70)$$

(with the expression for the entropy being unchanged). Obviously, the above procedure can be applied also to (2.61) and (2.63) as particular cases.

One can investigate also the other approach. According to the definition of a dense fluid, its energy density is a function of the two variables, ρ and T . The simplest possible functions with that property are those with the additive decomposition into a sum of two functions depending on single variables

$$E(\rho, T) = E_T(T) + E_\rho(\rho). \quad (2.71)$$

The Ansatz (2.71) can be inserted into (2.24):

$$\rho \frac{\partial}{\partial \rho} \left[\frac{E_T(T) + E_\rho(\rho)}{T} \right] + T \frac{\partial}{\partial T} \left[\frac{p(\rho, T)}{T\rho} \right] = 0, \quad (2.72)$$

what implies

$$\frac{\partial}{\partial T} \left[\frac{p(\rho, T)}{T\rho} \right] = -\frac{\rho}{T^2} \frac{\partial E_\rho(\rho)}{\partial \rho}. \quad (2.73)$$

The expression, standing in (2.73) under the symbol $\frac{\partial}{\partial T}$ is defined up to the addition of the arbitrary function of the variable ρ while the second part of the solution of (2.73) is the “indefinite integral” of the quantity standing on the r.h.s. of (2.73). As before, the operation of the indefinite integral with respect to the variable T is denoted Π_T , and the solution of (2.73) takes the following form

$$\frac{p(\rho, T)}{T\rho} + \Lambda(\rho) = \Pi_T \left[-\frac{\rho}{T^2} \frac{\partial E_\rho(\rho)}{\partial \rho} \right]. \quad (2.74)$$

In (2.74) the indefinite integral is taken with respect to the variable T and the variable ρ is a parameter; the dependence on ρ can be taken before the integral and the explicit form of the result is

$$\frac{p(\rho, T)}{T\rho} + \Lambda(\rho) = -\rho \frac{\partial E_\rho(\rho)}{\partial \rho} \Pi_T \left[\frac{1}{T^2} \right] = -\rho \frac{\partial E_\rho(\rho)}{\partial \rho} \left[-\frac{1}{T} + C \right]. \quad (2.75)$$

The r.h.s. of (2.75) can be written as the sum of two terms and then (2.75) takes a form

$$\frac{p(\rho, T)}{T\rho} + \Lambda(\rho) = \rho \frac{\partial E_\rho(\rho)}{T\partial \rho} - C\rho \frac{\partial E_\rho(\rho)}{\partial \rho}. \quad (2.76)$$

The symbol “ $\Lambda(\rho)$ ” means an arbitrary function of the variable ρ and therefore the expression $C\rho \frac{\partial E_\rho(\rho)}{\partial \rho}$ standing on the r.h.s. of (2.76) can be incorporated into $\Lambda(\rho)$;

$$\frac{p(\rho, T)}{T\rho} + \Lambda(\rho) = \rho \frac{\partial E_\rho(\rho)}{T\partial \rho}. \quad (2.77)$$

Now, both sides of (2.77) can be multiplied by $T\rho$ and the result is

$$p(\rho, T) + \rho\Lambda(\rho)T = \rho^2 \frac{\partial E_\rho(\rho)}{\partial \rho}. \quad (2.78)$$

In (2.78), the factor “ ρ ” can be incorporated into a symbol “ $\Lambda(\rho)$ ”, and the final result is

$$p(\rho, T) = \Lambda(\rho)T + \rho^2 \frac{\partial E_\rho(\rho)}{\partial \rho}. \quad (2.79)$$

One can see that the expression (2.79) is identical to the pressure given in (2.70).

It seems interesting to mention also the case of the Van der Waals gas; the expression for the pressure in the Van der Waals gas is of a general form

$$p(\rho, T) = f(\rho)T + \Phi(\rho), \quad (2.80)$$

that is, (2.80) is a sum of two functions; the first function is a product of an arbitrary function of the mass density multiplied by the temperature T and the second one is an arbitrary function of the mass density [19]. Similarly as before, (2.80) can be inserted into (2.24) and the result is

$$\rho \frac{\partial}{\partial \rho} \left[\frac{E(\rho, T)}{T} \right] + T \frac{\partial}{\partial T} \left[\frac{f(\rho)T + \Phi(\rho)}{T\rho} \right] = 0. \quad (2.81)$$

The identity (2.81) can be simplified to the form

$$\frac{\partial}{\partial \rho} \left[\frac{E(\rho, T)}{T} \right] = -\frac{T}{\rho} \frac{\partial}{\partial T} \left[\frac{\Phi(\rho)}{T\rho} \right]. \quad (2.82)$$

By analogy to the case (2.73), one can see that the expression standing in (2.82) under the symbol $\frac{\partial}{\partial \rho}$ is defined up to the addition of the arbitrary function of the variable T while the second part of the solution of (2.82) is the indefinite integral of the quantity standing on the r.h.s. of (2.82). As before, the operation of taking the indefinite integral with respect to the variable ρ is denoted as Π_ρ and the result is

$$\frac{E(\rho, T)}{T} + \Gamma(T) = \Pi_\rho \left\{ -\frac{T}{\rho} \frac{\partial}{\partial T} \left[\frac{\Phi(\rho)}{T\rho} \right] \right\}. \quad (2.83)$$

That identity can be simplified to the form

$$\frac{E(\rho, T)}{T} + \Gamma(T) = \frac{1}{T} \Pi_\rho \left\{ \frac{\Phi(\rho)}{\rho^2} \right\}. \quad (2.84)$$

After multiplying both sides of (2.84) by the temperature T and taking into account the definition of “ $\Gamma(T)$ ” one arrives at the identity

$$E(\rho, T) = \Gamma(T) + \Pi_\rho \left\{ \frac{\Phi(\rho)}{\rho^2} \right\}. \quad (2.85)$$

Now one can differentiate both sides of (2.85) with respect to the variable ρ and the result is

$$\frac{\partial}{\partial \rho} E(\rho, T) = \frac{\partial}{\partial \rho} \Pi_\rho \left\{ \frac{\Phi(\rho)}{\rho^2} \right\} = \frac{\Phi(\rho)}{\rho^2}. \quad (2.86)$$

It is interesting that our result is identical to (2.69) and to the model defined by the condition (2.71). In other words, different “independent” definitions result in the same model of a dense fluid.

In order to discuss the general case of a dense fluid satisfying Gibbs identity one can start with the identity (2.24)

$$\rho \frac{\partial}{\partial \rho} \left[\frac{E(\rho, T)}{T} \right] + T \frac{\partial}{\partial T} \left[\frac{p(\rho, T)}{T\rho} \right] = 0 \quad (2.87)$$

and its consequences

$$\frac{\partial}{\partial \rho} \left[\frac{E(\rho, T)}{T} \right] = -\frac{T}{\rho} \frac{\partial}{\partial T} \left[\frac{p(\rho, T)}{T\rho} \right] \quad (2.88)$$

and

$$\frac{\partial}{\partial T} \left[\frac{p(\rho, T)}{T\rho} \right] = -\frac{\rho}{T} \frac{\partial}{\partial \rho} \left[\frac{E(\rho, T)}{T} \right]. \quad (2.89)$$

Similarly as before, one can write

$$\frac{E(\rho, T)}{T} = \Gamma(T) + \Pi_\rho \left\{ -\frac{T}{\rho} \frac{\partial}{\partial T} \left[\frac{p(\rho, T)}{T\rho} \right] \right\} \quad (2.90)$$

and

$$\frac{p(\rho, T)}{T\rho} = \Lambda(\rho) + \Pi_T \left\{ -\frac{\rho}{T} \frac{\partial}{\partial \rho} \left[\frac{E(\rho, T)}{T} \right] \right\}. \quad (2.91)$$

The identity (2.90) can be written as

$$\frac{E(\rho, T)}{T} = \Gamma(T) - T \frac{\partial}{\partial T} \Pi_\rho \left[\frac{p(\rho, T)}{\rho^2 T} \right] \quad (2.92)$$

which after multiplying by the temperature takes the form

$$E(\rho, T) = \Gamma(T) - T^2 \frac{\partial}{\partial T} \Pi_\rho \left[\frac{p(\rho, T)}{\rho^2 T} \right]. \quad (2.93)$$

The expression (2.93) shows that for an arbitrary pressure $p(\rho, T)$ the energy density is defined up to arbitrary function $\Gamma(T)$, depending only on the temperature T . In order to stress that fact, for (2.93) we shall sometimes use a symbolic notation

$$E(\rho, T) = [\Gamma(T); p(\rho, T)]. \quad (2.94)$$

In turn, the identity (2.91) can be written in the form

$$\frac{p(\rho, T)}{T\rho} = \Lambda(\rho) - \rho \frac{\partial}{\partial \rho} \Pi_T \left[\frac{E(\rho, T)}{T^2} \right], \quad (2.95)$$

which after multiplying by $T\rho$ becomes

$$p(\rho, T) = \Lambda(\rho)T - \rho^2 T \frac{\partial}{\partial \rho} \Pi_T \left[\frac{E(\rho, T)}{T^2} \right]. \quad (2.96)$$

The expression (2.96) shows that for an arbitrary energy density $E(\rho, T)$ the pressure is defined up to arbitrary function $\Lambda(\rho)$ depending only on the mass density ρ and for (2.96) we shall sometimes use a symbolic notation

$$p(\rho, T) = [\Lambda(\rho); E(\rho, T)]. \quad (2.97)$$

Now the derivatives of the entropy $S(\rho, T)$ with respect to ρ and T can be determined by means of the expressions

$$\frac{\partial S(\rho, T)}{\partial \rho} = \frac{1}{T} \left[\frac{\partial E(\rho, T)}{\partial \rho} - \frac{p(\rho, T)}{\rho^2} \right] \quad (2.98)$$

and

$$\frac{\partial S(\rho, T)}{\partial T} = \frac{1}{T} \frac{\partial E(\rho, T)}{\partial T}, \quad (2.99)$$

which are consequences of the Gibbs identity.

According the literature ([18], see also p. 78 of [6]) a square of a sound speed in adiabatic conditions can be computed as a derivative of the pressure with respect to mass density, taken for a constant entropy. It is also well-known that similar derivative but taken for a constant temperature defines a square of a sound speed in isothermal conditions. In our discussion, the condition that both sound speeds are non-negative can be taken as the physical restriction imposed on admissible models (see [6]).

In our equations (2.93), (2.94) and (2.96), (2.97) the primitive fields are ρ and T and in order to compute the sound speed for adiabatic conditions one has to consider processes that take place for the constant entropy

$$S(\rho, T) = \text{const} = C. \quad (2.100)$$

According to the standard procedure (see, for example, [19]) that identity can be parameterized in terms of the mass density $\rho \rightarrow (\rho, T(\rho))$ and the result can

be differentiated to give

$$0 = \frac{d}{d\rho} S(\rho, T(\rho)) = \frac{\partial S(\rho, T)}{\partial \rho} + \frac{\partial S(\rho, T)}{\partial T} \frac{\partial T(\rho)}{\partial \rho}. \quad (2.101)$$

In turn, from (2.101) it is possible to determine $\frac{\partial T(\rho)}{\partial \rho}$ in terms of the derivatives of entropy with respect to ρ and T :

$$\frac{\partial T(\rho)}{\partial \rho} = - \frac{\frac{\partial S(\rho, T)}{\partial \rho}}{\frac{\partial S(\rho, T)}{\partial T}}, \quad (2.102)$$

which are given explicitly in (2.98) and (2.99). Now it is possible to write the formula for the square of sound speed in adiabatic conditions

$$\frac{d}{d\rho} p|_{S=\text{const}} = \frac{\partial p(\rho, T)}{\partial \rho} + \frac{\partial p(\rho, T)}{\partial T} \left[- \frac{\frac{\partial S(\rho, T)}{\partial \rho}}{\frac{\partial S(\rho, T)}{\partial T}} \right]. \quad (2.103)$$

Its alternative form is as follows

$$\begin{aligned} \frac{d}{d\rho} p|_{S=\text{const}} &= \frac{\partial p(\rho, T)}{\partial \rho} + \frac{\partial p(\rho, T)}{\partial T} \left\{ - \frac{\frac{1}{T} \left[\frac{\partial E(\rho, T)}{\partial \rho} - \frac{p(\rho, T)}{\rho^2} \right]}{\frac{1}{T} \frac{\partial E(\rho, T)}{\partial T}} \right\} = \\ &= \frac{\partial p(\rho, T)}{\partial \rho} - \frac{\partial p(\rho, T)}{\partial T} \left\{ \frac{\left[\frac{\partial E(\rho, T)}{\partial \rho} - \frac{p(\rho, T)}{\rho^2} \right]}{\frac{\partial E(\rho, T)}{\partial T}} \right\}. \end{aligned} \quad (2.104)$$

In order to discuss different models within our approach, it is possible to insert either (2.93) or (2.96) into (2.104). Let us start with a case of a generalized ideal gas (some remarks concerning the dense fluids shall be given later).

The entropy density for generalized ideal gas is given in (2.59)

$$S(\rho, T) = \int_{T_0}^T \frac{1}{T'} \frac{\partial E(T')}{\partial T'} dT' - \int_{\rho_0}^{\rho} \frac{p_0(\rho')}{\rho'^2} d\rho' \quad (2.105)$$

and its derivatives can be computed explicitly

$$\begin{aligned} \frac{\partial}{\partial \rho} S(\rho, T) &= \frac{\partial}{\partial \rho} \left\{ \int_{T_0}^T \frac{1}{T'} \frac{\partial E(T')}{\partial T'} dT' - \int_{\rho_0}^{\rho} \frac{p_0(\rho')}{\rho'^2} d\rho' \right\} = \\ &= \frac{\partial}{\partial \rho} \left\{ - \int_{\rho_0}^{\rho} \frac{p_0(\rho')}{\rho'^2} d\rho' \right\} = - \frac{p_0(\rho)}{\rho^2} \end{aligned} \quad (2.106)$$

and

$$\begin{aligned} \frac{\partial}{\partial T} S(\rho, T) &= \frac{\partial}{\partial T} \left\{ \int_{T_0}^T \frac{1}{T'} \frac{\partial E(T')}{\partial T'} dT' - \int_{\rho_0}^{\rho} \frac{p_0(\rho')}{\rho'^2} d\rho' \right\} = \\ &= \frac{\partial}{\partial T} \left\{ \int_{T_0}^T \frac{1}{T'} \frac{\partial E(T')}{\partial T'} dT' \right\} = \frac{1}{T} \frac{\partial E(T)}{\partial T}. \end{aligned} \quad (2.107)$$

For (2.106) and (2.107) the general expression (2.102) for $\frac{\partial T(\rho)}{\partial \rho}$ takes the following form

$$\frac{\partial T(\rho)}{\partial \rho} = - \frac{-\frac{p_0(\rho)}{\rho^2}}{\frac{1}{T} \frac{\partial E(T)}{\partial T}} = \frac{p_0(\rho)T}{\rho^2 \frac{\partial E(T)}{\partial T}}. \quad (2.108)$$

For the particular case of an ideal gas described by two parameters (2.61)

$$\frac{\partial E(T)}{\partial T} = \frac{\partial AT}{\partial T} = A, \quad p_0(\rho) = B\rho \quad (2.109)$$

and therefore

$$\frac{\partial T(\rho)}{\partial \rho} = \frac{BT}{A\rho}. \quad (2.110)$$

For the particular case of an ideal gas without the rotational and vibrational degrees of freedom (see (2.63), compare [19]) with

$$\rho E = \frac{3}{2}p, \quad A = \frac{3}{2}B, \quad (2.111)$$

(2.110) assumes the form

$$\frac{\partial T(\rho)}{\partial \rho} = \frac{BT}{A\rho} = \frac{BT}{\frac{3}{2}B\rho} = \frac{2T}{3\rho}. \quad (2.112)$$

Our result is consistent with [19], p. 101; the condition for the adiabatic transformation is the identity

$$T\rho^{-2/3} = \text{const} = C. \quad (2.113)$$

For an ideal gas the pressure is proportional to the product ρT and therefore the identity (2.113) can be equivalently written as

$$p\rho^{-5/3} = \text{const} = C \quad (2.114)$$

(compare [19], p. 101).

After inserting (2.58), (2.106), (2.107) and (2.108) into (2.103) one obtains

$$c_{adiab}^2 = \frac{d}{d\rho} p|_{S=\text{const}} = \frac{\partial}{\partial\rho} p_0(\rho)T + p_0(\rho) \frac{p_0(\rho)T}{\rho^2 \frac{\partial E(T)}{\partial T}}. \quad (2.115)$$

After inserting (2.109) into (2.15) one arrives at

$$c_{adiab}^2 = BT + \frac{B^2T}{A} = BT \left(1 + \frac{B}{A} \right). \quad (2.116)$$

After inserting (2.111) into (2.116) one can see that

$$c_{adiab}^2 = \frac{5}{3}BT. \quad (2.117)$$

In the above text, the equations for the “generalized ideal gas” have been inserted into the general expressions (2.103), (2.104). Also other models or the general expressions (2.93), (2.94) and (2.96), (2.97) can be inserted into the general expressions (2.103), (2.104) for the sound speed in adiabatic conditions.

The expression for the sound speed in isothermal conditions

$$c_{isoth}^2 = \frac{d}{d\rho} p|_{T=\text{const}} = \frac{\partial}{\partial\rho} \left\{ \Lambda(\rho)T - T\rho^2 \frac{\partial}{\partial\rho} \Pi_T \left[\frac{E(\rho, T)}{T^2} \right] \right\} =$$

$$T \frac{\partial \Lambda(\rho)}{\partial\rho} - T \frac{\partial}{\partial\rho} \left\{ \rho^2 \frac{\partial}{\partial\rho} \Pi_T \left[\frac{E(\rho, T)}{T^2} \right] \right\} \quad (2.118)$$

is simpler and therefore easier to interpret. In the second term of (2.118) the energy density $E(\rho, T)$ is differentiated with respect to the mass density:

$$T \frac{\partial}{\partial\rho} \left\{ \rho^2 \frac{\partial}{\partial\rho} \Pi_T \left[\frac{E(\rho, T)}{T^2} \right] \right\} = T \frac{\partial}{\partial\rho} \left\{ \rho^2 \Pi_T \left[\frac{1}{T^2} \frac{\partial}{\partial\rho} E(\rho, T) \right] \right\}. \quad (2.119)$$

Therefore, for the dense fluids this term can be different from zero but for the generalized ideal gases it vanishes and then the sound speed in isothermal conditions is a linear function of the temperature.

Moreover, even for some models of dense fluids the sound speed in isothermal conditions is a linear function of the temperature. An example here is the case (2.79)

$$p(\rho, T) = \Lambda(\rho)T + \rho^2 \frac{\partial E_\rho(\rho)}{\partial \rho} \quad (2.120)$$

that implies

$$\begin{aligned} c_{isoth}^2 &= \frac{d}{d\rho} p|_{T=\text{const}} = \\ &= \frac{\partial}{\partial \rho} \left\{ \Lambda(\rho)T + \rho^2 \frac{\partial E_\rho(\rho)}{\partial \rho} \right\} = \\ &= T \frac{\partial \Lambda(\rho)}{\partial \rho} + \frac{\partial}{\partial \rho} \left[\rho^2 \frac{\partial E_\rho(\rho)}{\partial \rho} \right]. \end{aligned} \quad (2.121)$$

It is worth to remember that the pressure is given by the expression (2.79) under the condition that the energy density is of a form (2.71). That form corresponds to a dense fluid but of a peculiar kind; the energy density can be decomposed into a sum of a function of the variable T and a function of the variable ρ . Therefore, a nonlinear temperature dependence of the sound velocity in isothermal conditions seems to suggest that the expression for the energy density contains “cross” terms (depending on the variables ρ and T simultaneously).

More detailed discussion would involve an application of virial expansions but that is outside the scope of this text.

It is worth to mention that the identity (2.18) can be derived from the Gibbs identity in terms of the differential forms: Gibbs identity

$$TdS = dE + pd \left[\frac{1}{\rho} \right]$$

implies

$$dS = \frac{1}{T}dE + \frac{p}{T}d \left[\frac{1}{\rho} \right] \quad (2.122)$$

and the condition that the entropy 1-form is closed can be written as the condition that the external derivative of dS vanishes

$$d^2S = 0. \quad (2.123)$$

After inserting (2.122) into (2.123) and taking ρ and T as primitive fields one obtains

$$\begin{aligned}
d \left[\frac{1}{T} dE + \frac{p}{T} d \left(\frac{1}{\rho} \right) \right] &= d \left[\frac{1}{T} \frac{\partial E}{\partial \rho} d\rho + \frac{1}{T} \frac{\partial E}{\partial T} dT - \frac{p}{\rho^2 T} d\rho \right] = \\
d \left\{ \left[\frac{1}{T} \frac{\partial E}{\partial \rho} - \frac{p}{\rho^2 T} \right] d\rho + d \left[\frac{1}{T} \frac{\partial E}{\partial T} dT \right] \right\} &= d \left\{ \left[\frac{1}{T} \frac{\partial E}{\partial \rho} - \frac{p}{\rho^2 T} \right] d\rho \right\} + \\
d \left[\frac{1}{T} \frac{\partial E}{\partial T} dT \right] &= d \left[\frac{1}{T} \frac{\partial E}{\partial \rho} - \frac{p}{\rho^2 T} \right] \wedge d\rho + d \left[\frac{1}{T} \frac{\partial E}{\partial T} \right] \wedge dT = \\
\frac{\partial}{\partial T} \left[\frac{1}{T} \frac{\partial E}{\partial \rho} - \frac{p}{\rho^2 T} \right] dT \wedge d\rho + \frac{\partial}{\partial \rho} \left[\frac{1}{T} \frac{\partial E}{\partial \rho} - \frac{p}{\rho^2 T} \right] d\rho \wedge d\rho + \\
\frac{\partial}{\partial T} \left[\frac{1}{T} \frac{\partial E}{\partial T} \right] dT \wedge dT + \frac{\partial}{\partial \rho} \left[\frac{1}{T} \frac{\partial E}{\partial T} \right] d\rho \wedge dT. & \quad (2.124)
\end{aligned}$$

After taking into account that

$$d\rho \wedge d\rho = 0, \quad dT \wedge dT = 0, \quad (2.125)$$

the identity (2.123) reduces to

$$0 = d^2 S = \frac{\partial}{\partial T} \left[\frac{1}{T} \frac{\partial E}{\partial \rho} - \frac{p}{\rho^2 T} \right] dT \wedge d\rho + \frac{\partial}{\partial \rho} \left[\frac{1}{T} \frac{\partial E}{\partial T} \right] d\rho \wedge dT. \quad (2.126)$$

Finally, the relation

$$dT \wedge d\rho = -d\rho \wedge dT \quad (2.127)$$

implies

$$\left\{ \frac{\partial}{\partial T} \left[\frac{1}{T} \frac{\partial E}{\partial \rho} - \frac{p}{\rho^2 T} \right] - \frac{\partial}{\partial \rho} \left[\frac{1}{T} \frac{\partial E}{\partial T} \right] \right\} dT \wedge d\rho = 0, \quad (2.128)$$

and after taking into account that an external product of the independent variables is different from zero

$$dT \wedge d\rho \neq 0 \quad (2.129)$$

one arrives at

$$\frac{\partial}{\partial T} \left[\frac{1}{T} \frac{\partial E}{\partial \rho} - \frac{p}{\rho^2 T} \right] - \frac{\partial}{\partial \rho} \left[\frac{1}{T} \frac{\partial E}{\partial T} \right] = 0 \quad (2.130)$$

what is identical to (2.18) (compare. [16], p. 269, formula (24)). In turn, after multiplying (2.130) with ρT , it is possible to obtain (2.24).

For completeness, it is worth to mention Wójcik’s observation that the quantity $\Omega(\rho, T)$ that satisfies the following identity

$$\frac{D}{Dt}\Omega(\rho, T) = \Phi(\rho, T)\frac{D}{Dt}E(\rho, T) + \Phi(\rho, T)p(\rho, T)\frac{D}{Dt}\frac{1}{\rho} \quad (2.131)$$

exists always by Pfaff’s theorem (see Whitham [18], p. 151) where $\Phi(\rho, T)$ is the corresponding integrating factor. Then the identity (2.131) can be used in a discussion of the Navier–Stokes–Fourier equations. Of course, such version of these equations is independent on the Gibbs identity (therefore the functions $E(\rho, T)$ and $p(\rho, T)$ are “uncorrelated”). After discussion of their solutions one could check whether the Gibbs identity is really necessary for modelling flow problems by means of the Navier–Stokes–Fourier equations.

In this text, only the problem of the existence of the entropy has been considered. In literature, one often discusses the existence of the entropy balance but that question is outside the scope of this text.

In the next chapter, the Navier–Stokes–Fourier equations are written in the invariant notation. In particular, the invariant derivative of the substantial form is used for further discussion of the relations between the Navier–Stokes–Fourier equations and thermostatics.

Chapter 3

The Navier–Stokes–Fourier equations and thermostatics. Invariant notation

Some relations concerning the invariant notation in fluid mechanics and kinetic theory have been given in Introduction. Now we shall discuss the differential operators applied in the invariant notation for Navier–Stokes–Fourier equations in more detail. Different “models” of the Galilean space-time are discussed in Appendices A,B,C and D. The fundamental relation between the standard formalism and the affine structure of the Galilean space-time is given by the inertial coordinates (see (1.5)):

$$R^4 \ni (t, x^\alpha) \rightarrow g + t\vec{w} + x^\alpha\vec{e}_\alpha \in G. \quad (3.1)$$

In (3.1), the point g of the Galilean space-time is added to the vectors $t\vec{w}$ and $x^\alpha\vec{e}_\alpha$; this operation is well-defined in literature on affine spaces ([1, 11, 20], see also Appendix A).

Now we shall repeat (with more details) the scheme sketched in [3] where the “affine differential quotients” have been applied together with the general procedure, described, for example, by Wintgen and Sulanke [22].

For a fixed coordinate system of the form (3.1) let us consider two different points corresponding to the coordinates (t, x^α) and (t', x^α) , correspondingly. According to the rules of affine geometry, it is possible to define the following quantity

$$\frac{(g + t'\vec{w} + x^i\vec{e}_i) - (g + t\vec{w} + x^i\vec{e}_i)}{t' - t} \quad (3.2)$$

and then

$$\begin{aligned} \frac{(g + t'\vec{w} + x^i\vec{e}_i) - (g + t\vec{w} + x^i\vec{e}_i)}{t' - t} &= \frac{(t'\vec{w} + x^i\vec{e}_i) - (t\vec{w} + x^i\vec{e}_i)}{t' - t} = \\ &= \frac{t'\vec{w} - t\vec{w}}{t' - t} = \frac{(t' - t)}{t' - t} = \vec{w}. \end{aligned} \quad (3.3)$$

One can see that (3.2) is an obvious analogue of a differential quotient. It is well-known that it is not possible to define a “distance” between non-simultaneous points of Galilean space-time. However, for finite-dimensional affine spaces it is possible to define the corresponding affine differential quotients because in finite-dimensional vector spaces all norms are equivalent (see [1, 11, 20] and Appendices A and B).

Therefore, it is possible to define the vector tangent to the “time coordinate t ” of the coordinate system (3.1) and after denoting it as $\frac{\vec{\partial}}{\partial t}$ one arrives at the following identity:

$$\frac{\vec{\partial}}{\partial t} = \lim_{t' \rightarrow t} \frac{(g_0 + t'\vec{w} + x^i\vec{e}_i) - (g_0 + t\vec{w} + x^i\vec{e}_i)}{t' - t} = \vec{w}. \quad (3.4)$$

In turn, after denoting the vector tangent to the coordinate x^1 as $\frac{\vec{\partial}}{\partial x^1}$ one obtains

$$\begin{aligned} \frac{\vec{\partial}}{\partial x^1} &= \\ \lim_{x^{1'} \rightarrow x^1} \frac{(g_0 + t\vec{w} + x^{1'}\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3) - (g_0 + t\vec{w} + x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3)}{x^{1'} - x^1} &= \\ &= \vec{e}_1, \end{aligned} \quad (3.5)$$

and correspondingly,

$$\frac{\vec{\partial}}{\partial x^2} = \vec{e}_2, \quad (3.6)$$

$$\frac{\vec{\partial}}{\partial x^3} = \vec{e}_3. \quad (3.7)$$

According to the standard approach of differential geometry (see, for example, [22]), the fields of dual base forms of the coordinate system (3.1) satisfy the defining relations

$$\left\langle Dt, \frac{\vec{\partial}}{\partial t} \right\rangle = \langle Dt, \vec{w} \rangle = 1, \quad (3.8)$$

$$\left\langle Dt, \overrightarrow{\frac{\partial}{\partial x^i}} \right\rangle = 0, \quad (3.9)$$

$$\left\langle Dx^j, \overrightarrow{\frac{\partial}{\partial t}} \right\rangle = \langle Dx^j, \overrightarrow{w} \rangle = 0, \quad (3.10)$$

$$\left\langle Dx^j, \overrightarrow{\frac{\partial}{\partial x^i}} \right\rangle = \langle Dx^j, \overrightarrow{e_i} \rangle = \delta_i^j, \quad (3.11)$$

with δ_i^j being the Kronecker delta.

Of course, the vectors $\{\overrightarrow{w}, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$ form a basis in T_G and the forms $\{Dt, Dx^1, Dx^2, Dx^3\}$ form a basis in T_G^* .

A complete derivative of the scalar function $\Phi(g)$ can be explicitly computed in any affine coordinate system on G after using the general definition (Schwartz [11], see also Appendix B) to the inertial coordinates (3.1);

$$D\Phi(g) = \frac{\partial\Phi(g + t\overrightarrow{w} + x^\alpha\overrightarrow{e_\alpha})}{\partial t} Dt + \frac{\partial\Phi(g + t\overrightarrow{w} + x^\alpha\overrightarrow{e_\alpha})}{\partial x^j} Dx^j. \quad (3.12)$$

A complete derivative of a vector field on G with values in T_G can be computed in any affine coordinate system on G according to [11] (see also Appendix B). Let us denote such vector field as $\overrightarrow{B(g)}$; it can be decomposed in the basis $\{\overrightarrow{w}, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$ in T_G

$$\overrightarrow{B(g)} = B^t(g)\overrightarrow{w} + B^i(g)\overrightarrow{e_i}. \quad (3.13)$$

A complete derivative of (3.13) can be defined in the following way

$$D\overrightarrow{B(g)} = \overrightarrow{w} \otimes DB^t(g) + \overrightarrow{e_i} \otimes DB^i(g). \quad (3.14)$$

The coordinates of the vector fields $\overrightarrow{B(g)}$ in a fixed basis $\{\overrightarrow{w}, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$ are the real functions on G and therefore its complete derivatives can be computed in the inertial coordinates according to the rule (3.12);

$$\begin{aligned} D\overrightarrow{B(g)} &= \overrightarrow{w} \otimes DB^t(g) + B^i(g)\overrightarrow{e_i} \otimes DB^i(g) = \\ &= \overrightarrow{w} \otimes \left[\frac{\partial B^t(g + t\overrightarrow{w} + x^\alpha\overrightarrow{e_\alpha})}{\partial t} Dt + \frac{\partial B^t(g + t\overrightarrow{w} + x^\alpha\overrightarrow{e_\alpha})}{\partial x^j} Dx^j \right] + \\ &+ \overrightarrow{e_i} \otimes \left[\frac{\partial B^i(g + t\overrightarrow{w} + x^\alpha\overrightarrow{e_\alpha})}{\partial t} Dt + \frac{\partial B^i(g + t\overrightarrow{w} + x^\alpha\overrightarrow{e_\alpha})}{\partial x^j} Dx^j \right] = \\ &= \overrightarrow{w} \otimes \frac{\partial B^t(g + t\overrightarrow{w} + x^\alpha\overrightarrow{e_\alpha})}{\partial t} Dt + \overrightarrow{w} \otimes \frac{\partial B^t(g + t\overrightarrow{w} + x^\alpha\overrightarrow{e_\alpha})}{\partial x^j} Dx^j + \end{aligned}$$

$$\vec{e}_i \otimes \frac{\partial B^i (g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial t} Dt + \vec{e}_i \otimes \frac{\partial B^i (g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial x^j} Dx^j. \quad (3.15)$$

One can introduce the “four-dimensional” divergence of a vector field in a four-dimensional affine space (in terms of a corresponding contraction) and in general affine coordinates it is defined in [11] (see also [1] and Appendix B). In order to compute it in the inertial coordinates one can make use of the linearity of a contraction (the operation of taking a trace of a tensor is denoted as “ Tr ”):

$$\begin{aligned} Div \overrightarrow{B}(g) &= Tr \left\{ D\overrightarrow{B}(g) \right\} = \langle DB^t(g), \vec{w} \rangle + \langle DB^i(g), \vec{e}_i \rangle = \\ &\left\langle \frac{\partial B^t (g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial t} Dt, \vec{w} \right\rangle + \left\langle \frac{\partial B^t (g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial x^j} Dx^j, \vec{w} \right\rangle + \\ &\left\langle \frac{\partial B^i (g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial t} Dt, \vec{e}_i \right\rangle + \left\langle \frac{\partial B^i (g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial x^j} Dx^j, \vec{e}_i \right\rangle = \\ &\frac{\partial B^t (g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial t} \langle Dt, \vec{w} \rangle + \frac{\partial B^t (g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial x^j} \langle Dx^j, \vec{w} \rangle + \\ &\frac{\partial B^i (g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial t} \langle Dt, \vec{e}_i \rangle + \frac{\partial B^i (g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial x^j} \langle Dx^j, \vec{e}_i \rangle. \end{aligned} \quad (3.16)$$

In order to get the explicit form of $Div \overrightarrow{B}(g)$ one can use the “duality conditions” (3.8)–(3.11):

$$\begin{aligned} Div \overrightarrow{B}(g) &= Tr \left\{ D\overrightarrow{B}(g) \right\} = \\ &\frac{\partial B^t (g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial t} + \frac{\partial B^i (g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial x^j} \delta_i^j = \\ &\frac{\partial B^t (g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial t} + \frac{\partial B^i (g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial x^i}. \end{aligned} \quad (3.17)$$

It can be easily seen that if in inertial coordinates $B^t(g) = 0$ then the “four-dimensional” complete derivative of $\overrightarrow{B}(g)$ belongs to $S \otimes T_G^*$. In order to write fluid mechanics in an invariant manner one has to compute the “four-dimensional” complete derivative of the “non-relativistic four-velocity” $\overrightarrow{c}(g)$. In Appendix D the similar problem is discussed for the more general case of the “Galilean space-time with measurable time intervals” and the conclusions remain valid also now but it is instructive to make explicit calculations in inertial coordinate systems. In Introduction, the non-relativistic four-velocity has been written in the basis $\{\vec{w}, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ as

$$G \ni g \rightarrow \overrightarrow{c}(g) = \vec{w} + u^\alpha(g) \vec{e}_\alpha \in W \quad (3.18)$$

(see (1.23)) and after comparing (3.18) with (3.13) one can see that (3.18) can be written in the form (3.13) under the condition that

$$B^t(g) = 1, \quad B^i(g) = u^\alpha(g). \quad (3.19)$$

Since a complete derivative of a constant function is equal to zero, from (3.14) one can obtain the explicit form of the “four-dimensional” complete derivative of $\overrightarrow{c(g)}$

$$D\overrightarrow{c(g)} = \vec{e}_i \otimes Du^i(g) \in S \otimes T_G^*. \quad (3.20)$$

As it has been already mentioned, in [3] it has been observed that the invariant counterpart of a “substantial derivative” is a directional derivative along the non-relativistic four velocity. In general, the directional derivatives can exist in cases when a complete derivative do not exist [11] but for “smooth” fields the directional derivative can be defined as a contraction of a complete derivative with a vector field. In Appendix B, this contraction is written explicitly but now we shall can use for contraction the symbol “ \odot ”. We shall discuss the case of a substantial derivative of a scalar quantity first and since the complete derivative of a scalar quantity is given as (3.12) the corresponding contraction is:

$$\begin{aligned} \vec{c}(g) \odot D\Phi(g) &= \vec{c}(g) \odot \left[\frac{\Phi(g)}{\partial t} Dt + \frac{\Phi(g)}{\partial x^j} Dx^j \right] = \\ &= \vec{c}(g) \odot \frac{\Phi(g)}{\partial t} Dt + \vec{c}(g) \odot \frac{\Phi(g)}{\partial x^j} Dx^j = \\ &= \frac{\Phi(g)}{\partial t} \langle Dt, \vec{c}(g) \rangle + \frac{\Phi(g)}{\partial x^j} \langle Dx^j, \vec{c}(g) \rangle. \end{aligned} \quad (3.21)$$

Till now, only the linearity of contraction has been used. Now we insert into (3.21) the explicit expression for the non-relativistic four-velocity (3.18) and make use of the duality relations (3.6)–(3.9):

$$\begin{aligned} \vec{c}(g) \odot D\Phi(g) &= \frac{\Phi(g)}{\partial t} \langle Dt, \vec{w} + u(g)^\alpha \vec{e}_\alpha \rangle + \frac{\Phi(g)}{\partial x^j} \langle Dx^j, \vec{w} + u(g)^\alpha \vec{e}_\alpha \rangle = \\ &= \frac{\Phi(g)}{\partial t} \langle Dt, \vec{w} \rangle + \frac{\Phi(g)}{\partial t} \langle Dt, u(g)^\alpha \vec{e}_\alpha \rangle + \frac{\Phi(g)}{\partial x^j} \langle Dx^j, \vec{w} \rangle \\ &\quad + \frac{\Phi(g)}{\partial x^j} \langle Dx^j, u(g)^\alpha \vec{e}_\alpha \rangle = \frac{\Phi(g)}{\partial t} \langle Dt, \vec{w} \rangle + \\ &+ u(g)^\alpha \frac{\Phi(g)}{\partial t} \langle Dt, \vec{e}_\alpha \rangle + \frac{\Phi(g)}{\partial x^j} \langle Dx^j, \vec{w} \rangle + u(g)^\alpha \frac{\Phi(g)}{\partial x^j} \langle Dx^j, \vec{e}_\alpha \rangle = \\ &= \frac{\Phi(g)}{\partial t} + u(g)^\alpha \frac{\Phi(g)}{\partial x^j} \delta^j_\alpha = \frac{\Phi(g)}{\partial t} + u(g)^j \frac{\Phi(g)}{\partial x^j} \end{aligned} \quad (3.22)$$

The above formula shows that a directional derivative of a scalar variable along the non-relativistic four velocity is identical to the “substantial derivative” of a scalar variable. In order to compute the directional derivative of a vector field (3.13) it is sufficient to compute the corresponding contraction of $\vec{c}(g)$ and (3.14)–(3.15):

$$\begin{aligned} \vec{c}(g) \odot \overrightarrow{DB(g)} &= \vec{w} \langle DB^t(g), \vec{c}(g) \rangle + \vec{e}_i \langle DB^i(g), \vec{c}(g) \rangle = \\ & \vec{w} \frac{\partial B^t(g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial t} \langle Dt, \vec{c}(g) \rangle + \\ & \vec{w} \frac{\partial B^t(g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial x^j} \langle Dx^j, \vec{c}(g) \rangle + \\ & \vec{e}_i \frac{\partial B^i(g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial t} \langle Dt, \vec{c}(g) \rangle + \\ & \vec{e}_i \frac{\partial B^i(g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial x^j} \langle Dx^j, \vec{c}(g) \rangle. \end{aligned} \quad (3.23)$$

On account of (3.18), it is easy to see that

$$\langle Dt, \overrightarrow{c(g)} \rangle = \langle Dt, \vec{w} + u^\alpha(g) \vec{e}_\alpha \rangle = 1 \quad (3.24)$$

and

$$\langle Dx^j, \vec{c}(g) \rangle = u^\alpha(g) \langle Dx^j, \vec{e}_\alpha \rangle = u^\alpha(g) \delta^j_\alpha. \quad (3.25)$$

After inserting (3.24) and (3.25) into (3.23) one arrives finally at

$$\begin{aligned} \vec{c}(g) \odot \overrightarrow{DB(g)} &= \\ & \vec{w} \frac{\partial B^t(g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial t} + \vec{w} \frac{\partial B^t(g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial x^j} u^\alpha(g) \delta^j_\alpha + \\ & \vec{e}_i \frac{\partial B^i(g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial t} + \vec{e}_i \frac{\partial B^i(g + t\vec{w} + x^\alpha \vec{e}_\alpha)}{\partial x^j} u^\alpha(g) \delta^j_\alpha. \end{aligned} \quad (3.26)$$

A particular case of the expression (3.26) is a contraction $\overrightarrow{c(g)} \odot \overrightarrow{Dc(g)}$:

$$\overrightarrow{c(g)} \odot \overrightarrow{Dc(g)} = \vec{e}_i \frac{\partial u^i(g)}{\partial t} + \vec{e}_i u^j(g) \frac{\partial u^i(g)}{\partial x^j}. \quad (3.27)$$

One can see that (3.27) is equal to the standard “substantial derivative” of the velocity $u^i(g) \vec{e}_i$.

According to notational convention applied by Schwartz [11] the directional derivative taken with respect to the vector field $\overrightarrow{B(g)}$ should be denoted as $D_{\overrightarrow{B(g)}}$ but since we shall apply the directional derivatives along $\overrightarrow{c(g)}$ only it is useful to apply notational identification

$$D_{\overrightarrow{c(g)}} = \frac{D}{Dt}. \quad (3.28)$$

In the standard notation, the mass balance in fluid mechanics has the following form

$$\frac{\partial \rho}{\partial t} + u^i(g) \frac{\partial \rho}{\partial x^i} + \rho \frac{\partial u^i(g)}{\partial x^i} = 0 \quad (3.29)$$

where ρ denotes the mass density. Our discussion shows that the invariant counterpart of the mass balance is

$$\frac{D}{Dt} \rho(g) + \rho \text{Div} \{ \vec{c}(g) \} = 0. \quad (3.30)$$

The remaining balance laws require some additional discussion. The important property of the models discussed in Appendices C and D is that they possess two complete derivatives; the “four-dimensional” complete derivative “ D ” and the “three-dimensional” complete derivative “ ∇ ”. Now a “complete” Galilean space-time is considered and here that effect is also present and the “three-dimensional” complete derivative becomes the “Euclidean gradient” (on the spaces of simultaneous events) and shall be denoted in a standard manner as “ ∇ ”. One can easily define “spatial gradients” with respect to the operator ∇ of the “Euclidean” quantities, (like, for example, “ $\nabla \rho(g)$ ”) but in order to define spatial gradients of non-euclidian quantities (like, for example, “ $\nabla \vec{c}(g)$ ”) one can make use of the more general definitions discussed in [1, 11] (see also Appendix A and B). A rigorous definition of the meaning of $\nabla \vec{c}(g)$ can be obtained as a special case of a complete derivative of a mapping between finite-dimensional affine spaces (alternatively, it can be considered as a particular case of a mapping between finite-dimensional manifolds).

However, now we want to discuss different aspects of the inertial coordinates and therefore we shall discuss the transition from the inertial coordinates on G to the corresponding coordinates on the given space of the simultaneous events $H_{[g]}$ (by definition, this space consists of the “events” simultaneous to the event $g \in G$). Let us consider the inertial coordinate system of the following form

$$(t, x^i) \rightarrow g + t\vec{w} + x^i \vec{e}_i \in H_{[g]+t} \quad (3.31)$$

The time coordinate “ t ” in (3.31) can be put equal to zero and in such a case one obtains an affine coordinate system on $H_{[g]}$:

$$(x^1, x^2, x^3) \rightarrow g + 0\vec{w} + x^i\vec{e}_i = g + x^i\vec{e}_i \in H_{[g]}. \quad (3.32)$$

For any vector field $\overrightarrow{B(g)}$ on G with the values in T_G one can define the quantity $\overrightarrow{\nabla B(g)}$ in the following manner

$$\overrightarrow{\nabla B(g)} = \frac{\overrightarrow{\partial B(g + x^i\vec{e}_i)}}{\partial x^j} \otimes dx^j \quad (3.33)$$

where dx^j satisfy the following “three-dimensional” duality conditions

$$\langle dx^j, \vec{e}_i \rangle = \delta_i^j, \quad i, j = 1, 2, 3. \quad (3.34)$$

The additional property of orthonormal coordinates is that “three-dimensional” functionals dx^j can be represented as “taking the scalar product” according to the identification

$$dx^j = (\vec{e}_j, \cdot). \quad (3.35)$$

It is obvious that for $\overrightarrow{B(g)}$ taking the values in T_G the two-point tensor field $\overrightarrow{\nabla B(g)}$ takes the values in $T_G \otimes S^*$. In particular, one can compute $\overrightarrow{\nabla \vec{c}(g)}$ and from the definition (3.33) and from (3.18) one arrives at

$$\begin{aligned} \overrightarrow{\nabla \vec{c}(g)} &= \frac{\partial \vec{c}(g + x^i\vec{e}_i)}{\partial x^j} \otimes dx^j = \frac{\partial [\vec{w} + u^\alpha(g)\vec{e}_\alpha]}{\partial x^j} \otimes dx^j = \\ &\vec{e}_\alpha \frac{\partial u^\alpha(g)}{\partial x^j} \otimes dx^j \end{aligned} \quad (3.36)$$

what shows that $\overrightarrow{\nabla \vec{c}(g)}$ takes the values in $S \otimes S^*$. Therefore it is possible to define contraction of $\overrightarrow{\nabla \vec{c}(g)}$ and the corresponding result is

$$\begin{aligned} Tr \overrightarrow{\nabla \vec{c}(g)} &= Tr \left\{ \vec{e}_\alpha \frac{\partial u^\alpha(g)}{\partial x^j} \otimes dx^j \right\} = \\ \frac{\partial u^\alpha(g)}{\partial x^j} \langle dx^j, \vec{e}_\alpha \rangle &= \frac{\partial u^\alpha(g)}{\partial x^j} \delta_\alpha^j = \frac{\partial u^j(g)}{\partial x^j}. \end{aligned} \quad (3.37)$$

From the identity (3.37) one can see that the trace of $\overrightarrow{\nabla \vec{c}(g)}$ is equal to the “standard divergence” of velocity and therefore it can be denoted as “ $div \vec{c}(g)$ ”

$$div \vec{c}(g) = Tr \{ \overrightarrow{\nabla \vec{c}(g)} \} = \frac{\partial u^j(g)}{\partial x^j}. \quad (3.38)$$

Moreover, from (3.16) and (3.20) it is possible to compute the “four-dimensional” divergence of $\vec{c}(g)$ and it can be easily checked that both divergencies are equal, that is

$$\operatorname{div}\vec{c}(g) = \operatorname{Tr} \{ \nabla \vec{c}(g) \} = \operatorname{Div} \{ \vec{c}(g) \} = \operatorname{Tr} \{ D\vec{c}(g) \}. \quad (3.39)$$

These identities show that the non-relativistic four-velocity satisfies many identities and those identities are necessary to maintain equivalence of the invariant formalism and the “standard” approach.

In terms of the differential operators described in this chapter it is possible to write the invariant form of the Navier–Stokes–Fourier equations

$$\frac{D}{Dt} \rho(g) + \rho \operatorname{Div} \{ \vec{c}(g) \} = 0, \quad (3.40)$$

$$\rho \frac{D}{Dt} \vec{c}(g) = \operatorname{div} \hat{T}(g), \quad (3.41)$$

$$\rho \frac{D}{Dt} E = \hat{T}(g) : \nabla c - \nabla q, \quad (3.42)$$

where $\hat{T}(g)$ denotes the (symmetric) Cauchy stress tensor, E denotes the energy density (per unit mass) and “:” means a double contraction. The Cauchy stress tensor $\hat{T}(g)$ is a field on G taking the values in $S \otimes S$ and the heat flux the q is a vector field on G taking the values in S . Usually $\hat{T}(g)$ is defined as a function of other variables and the example here is (2.1), (2.6).

In turn, the “Gibbs relation” in [16] has been written in terms of the substantial derivatives and now it is possible to see another aspect of this relation: let us consider the mass density $\rho(g)$ and the temperature $T(g)$ as the real functions on G :

$$G \ni g \rightarrow \rho(g) \in R, \quad (3.43)$$

$$G \ni g \rightarrow T(g) \in R. \quad (3.44)$$

According the definition (3.12), it is possible to define complete derivatives of $\rho(g)$ and $T(g)$;

$$\rho(g) \rightarrow D\rho(g) \in T_G^*, \quad (3.45)$$

$$T(g) \rightarrow DT(g) \in T_G^*. \quad (3.46)$$

The energy density (per unit mass) $E(\rho, T)$ and the pressure $p(\rho, T)$ can be understood as the composite functions on the space-time

$$G \ni g \rightarrow E(\rho(g), T(g)) \in R \quad (3.47)$$

$$G \ni g \rightarrow p(\rho(g), T(g)) \in R. \quad (3.48)$$

Now, one can compute the complete derivatives of $E(\rho, T)$ and $p(\rho, T)$

$$E(\rho(g), T(g)) \rightarrow DE(\rho(g), T(g)) \quad (3.49)$$

$$p(\rho(g), T(g)) \rightarrow Dp(\rho(g), T(g)). \quad (3.50)$$

It is possible to ask about the existence of such real function $S(\rho, T)$ that for all fields $\rho(g)$ and $T(g)$ the following identity holds:

$$\begin{aligned} T(g)DS(\rho(g), T(g)) &= DE(\rho(g), T(g)) + p(\rho(g), T(g)) D \begin{bmatrix} 1 \\ \rho \end{bmatrix} = \\ &DE(\rho(g), T(g)) - \frac{p(\rho(g), T(g))}{\rho^2} D\rho. \end{aligned} \quad (3.51)$$

Let $\overrightarrow{c(g)}$ denote the non-relativistic four-velocity field on G ; in Schwartz's notation [11] the result of a contraction of $\overrightarrow{c(g)}$ and (3.51) is

$$\begin{aligned} T(g)D_{\overrightarrow{c(g)}}S(\rho(g), T(g)) &= D_{\overrightarrow{c(g)}}E(\rho(g), T(g)) + p(\rho(g), T(g)) D_{\overrightarrow{c(g)}} \begin{bmatrix} 1 \\ \rho \end{bmatrix} = \\ &D_{\overrightarrow{c(g)}}E(\rho(g), T(g)) - \frac{p(\rho(g), T(g))}{\rho^2} D_{\overrightarrow{c(g)}}\rho, \end{aligned} \quad (3.52)$$

and in our notational convention (3.28) the identity (3.52) takes a form

$$\begin{aligned} T(g)\frac{D}{Dt}S(\rho(g), T(g)) &= \frac{D}{Dt}E(\rho(g), T(g)) + p(\rho(g), T(g)) \frac{D}{Dt} \begin{bmatrix} 1 \\ \rho \end{bmatrix} = \\ &\frac{D}{Dt}E(\rho(g), T(g)) - \frac{p(\rho(g), T(g))}{\rho^2} \frac{D}{Dt}\rho, \end{aligned} \quad (3.53)$$

identical to that applied in [16, 17].

Chapter 4

Invariant formulation of the non-relativistic distribution function

As it has been already mentioned in Introduction, the “invariant” counterpart of the non-relativistic distribution function is a non-negative function on the product $G \times W$. In turn, the invariant counterpart of “ μ -space” is a product $H_{[g]} \times W$. On p. 8 of Introduction, the inertial coordinate systems on the product $G \times W$ have been introduced. In this chapter, the “invariant” locally equilibrium distribution functions are introduced. For locally equilibrium distribution functions, elementary moment expressions are written in a coordinate-free manner.

According to Introduction, the differences of elements from W are spatial. In particular, from arbitrary element $\vec{w} \in W$ it is possible to subtract the value of a four-velocity field $\vec{c}(g)$ and the resulting difference is spatial:

$$g \in G, \quad \vec{w} \in W \Rightarrow \vec{w} - \vec{c}(g) \in S. \quad (4.1)$$

In turn, on S there exists a scalar product (see (1.1), (1.2)). Therefore, for a given four-velocity field $\vec{c}(g)$, it is possible to define a non-negative function on the product $G \times W$ after taking a scalar product of (4.1) with itself:

$$G \times W \ni (g, \vec{w}) \rightarrow \left[\vec{w} - \vec{c}(g) \right] \cdot \left[\vec{w} - \vec{c}(g) \right] \geq 0. \quad (4.2)$$

Let $C(g)$ and $A(g)$ be two real functions on the Galilean space-time G . Then it is possible to define the following function

$$f_0(g, w) = C(g) \exp \left\{ -A(g) \left[\vec{w} - \vec{c}(g) \right] \cdot \left[\vec{w} - \vec{c}(g) \right] \right\}. \quad (4.3)$$

Let \vec{w}_I denote the arbitrary element of W ; then among all possible bases in T_G it is possible to distinguish such basis $\{\vec{w}_I, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$, that $\vec{w}_I \in W$ and $\vec{e}_i \in S$, $i = 1, 2, 3$, with the additional property that

$$(\vec{e}_i, \vec{e}_i) = \delta_{\alpha\beta}, \quad (4.4)$$

where $\alpha, \beta = 1, 2, 3$ and $\delta_{\alpha\beta}$ is the Kronecker's symbol.

Alternatively, it is possible to define a “moving reper” on G given as $\{\vec{c}(g), \vec{e}_1, \vec{e}_2, \vec{e}_3\}$, $\vec{e}_i \in S$, $i = 1, 2, 3$. Now, the “moving” leg $\vec{c}(g)$ of the reper $\{\vec{c}(g), \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ can be expressed in a “constant reper” according to the identity

$$\vec{c}(g) = \vec{w}_I + [\vec{c}(g) - \vec{w}_I] = \vec{w}_I + c_I^i(g) \vec{e}_i, \quad (4.5)$$

what can be written equivalently as

$$\vec{c}(g) = \vec{w}_I + \overline{c}_I(g), \quad (4.6)$$

where $\overline{c}_I(g)$ is a corresponding field of “spatial” vectors on G . Similar decomposition can be done for the “molecular four-velocity”

$$\vec{w} = \vec{w}_I + \vec{u}_I, \quad (4.7)$$

where \vec{u}_I is also spatial. We repeat here the notation from Introduction (compare (1.26)–(1.32)) but with a minor modification. The reason for this is that effects discussed here are a bit more subtle because it is necessary to distinguish “molecular four-velocity”, macroscopic four-velocity, molecular velocity for a fixed inertial coordinate system and macroscopic velocity for a fixed coordinate system.

After inserting (4.6) and (4.7) into (4.1) one arrives at

$$\vec{w} - \vec{c}(g) = [\vec{w}_I + \vec{u}_I] - [\vec{w}_I + \overline{c}_I(g)] = \vec{u}_I - \overline{c}_I(g). \quad (4.8)$$

After inserting (4.8) into (4.2) one arrives at

$$\begin{aligned} C(g) \exp \left\{ -A(g) \left[\vec{w} - \vec{c}(g) \right] \cdot \left[\vec{w} - \vec{c}(g) \right] \right\} = \\ C(g) \exp \left\{ -A(g) \left[\vec{u}_I - \overline{c}_I(g) \right] \cdot \left[\vec{u}_I - \overline{c}_I(g) \right] \right\}. \end{aligned} \quad (4.9)$$

The expression (4.9) is consistent with the locally-equilibrium Maxwell–Boltzmann distribution function, given in Huang's monograph [19]. The only difference

is that (4.9) explicitly contains the index “ I ” informing about the non-relativistic four-velocity \vec{w}_I of the inertial observer.

In general, moment identities can be written either “in a fixed basis” or in a “moving reper” of the kind described above. For illustration, let us compute elementary “invariant” moments of the locally equilibrium distribution functions. We already know that the expression (1.29) describes the local density of molecules and now we shall compute that quantity for the locally-equilibrium distribution function (4.3). In order to stress the fact that the local density of molecules is computed with respect to the locally equilibrium distribution function we shall introduce the additional index “0”:

$$n_0(g) = \int_W C(g) \exp \{-A(g) [\vec{w} - \vec{c}(g)] \cdot [\vec{w} - \vec{c}(g)]\} d^3 \vec{w}. \quad (4.10)$$

In order to compute (4.10), let us change the variables according to the formula

$$\vec{w} - \vec{c}(g) = \vec{u} \in S. \quad (4.11)$$

From the definition of the Galilean space-time one knows that S is an Euclidean space and therefore it is endowed with an Euclidean volume measure. As it has been already described in Introduction, instead $\vec{w} - \vec{c}(g)$ one can put the euclidian vector \vec{u} and in place of $d^3 \vec{w}$ it is possible to insert $d^3 [\vec{u} - \vec{c}(g)] = d^3 \vec{u}$. Therefore

$$\begin{aligned} n_0(g) &= C(g) \int_W \exp \{-A(g) [\vec{w} - \vec{c}(g)] \cdot [\vec{w} - \vec{c}(g)]\} d^3 \vec{w} = \\ &= C(g) \int_S \exp [-A(g) \vec{u} \cdot \vec{u}] d^3 \vec{u}. \end{aligned} \quad (4.12)$$

The explicit form of (4.12) can be taken from p. 71 of [19]

$$n_0(g) = C(g) \left(\frac{\pi}{A(g)} \right)^{\frac{3}{2}}$$

from which Huang concludes that $A(g) > 0$ and

$$C(g) = \left(\frac{A(g)}{\pi} \right)^{\frac{3}{2}} n_0(g). \quad (4.13)$$

The average energy ϵ of a molecule is defined on p. 71 of [19] and in “invariant” notation its form is

$$\epsilon(g) = \frac{\int_W \frac{1}{2} m [\vec{w} - \vec{c}(g)] \cdot [\vec{w} - \vec{c}(g)] C(g) \exp \{-A(g) [\vec{w} - \vec{c}(g)] \cdot [\vec{w} - \vec{c}(g)]\} d^3 \vec{w}}{\int_W C(g) \exp \{-A(g) [\vec{w} - \vec{c}(g)] \cdot [\vec{w} - \vec{c}(g)]\} d^3 \vec{w}}, \quad (4.14)$$

where m is the mass of a molecule. Then, according to Huang,

$$\epsilon(g) = \frac{3}{4} \frac{m}{A(g)}$$

and

$$A(g) = \frac{3}{4} \frac{m}{\epsilon(g)}, \quad (4.15)$$

where m denotes the mass of a molecule. After substituting (4.15) into (4.13) one obtains for the parameter $C(g)$ the expression

$$C(g) = n(g) \left(\frac{3m}{4\pi\epsilon(g)} \right)^{\frac{3}{2}}. \quad (4.16)$$

On pp. 71 and 72 Huang introduces the equation of state (basing on elementary considerations); it relates the pressure with the local density of molecules and the average energy of a molecule

$$P(g) = \frac{2}{3} n(g) \epsilon(g). \quad (4.17)$$

Then Huang introduces the “experimental” definition of the temperature T by

$$P = nk_B T,$$

where k_B is Boltzmann’s constant. Hence

$$\epsilon(g) = \frac{3}{2} k_B T(g) \quad (4.18)$$

and finally our distribution function (4.3) can be parameterized by “physical” parameters:

$$f_0 \left(n(g), T(g), \vec{c}(g); \vec{w} \right) = n(g) \left(\frac{m}{2\pi k_B T(g)} \right)^{\frac{3}{2}} \exp \left\{ - \frac{m [\vec{w} - \vec{c}(g)] \cdot [\vec{w} - \vec{c}(g)]}{2k_B T(g)} \right\}. \quad (4.19)$$

In general, the moment expressions can be computed with respect to the arbitrary distribution function $f(g, \vec{w})$. In standard formalism, a general form of moment identities for Boltzmann equation was published by Banach and Piekarski in 1989 (see [56]). In invariant formulation, one can define a class of invariant moments by means of powers of the non-relativistic four-velocity:

$$T^n(g) = \int_W \otimes^n \vec{w} f(g, \vec{w}) d^3 \vec{w}, \quad (4.20)$$

$\vec{w} \in W$. From the definition of the chronological form it follows that after taking contraction of the chronological form Ψ with $T^n(g)$ one arrives at $T^{n-1}(g)$; it is a direct consequence of the definition of the set W (compare (1.3)). From the symmetry of the moment expression (4.20) one can see that the contraction of Ψ and $T^n(g)$ does not depend on the choice of a contracted index of $T^n(g)$ and can be computed in the following way:

$$\begin{aligned} Tr \Psi \otimes T^n(g) &= \Psi \odot T^n(g) = \int_W \otimes^{n-1} \vec{w} \langle \Psi, \vec{w} \rangle f(g, \vec{w}) d^3 \vec{w} = \\ &= \int_W \otimes^{n-1} \vec{w} \cdot 1 f(g, \vec{w}) d^3 \vec{w} = T^{n-1}(g), \end{aligned} \quad (4.24)$$

where the definition (1.3) has been applied and

$$T^0(g) = \int_W f(g, \vec{w}) d^3 \vec{w}. \quad (4.22)$$

We shall see later that the moment expressions for T^1 and T^0 are of special importance because from T^1 and T^0 one can determine the moment expression for the four-velocity of monoatomic gas. Now we shall check it for the particular case of the locally-equilibrium distribution function; let us compute T^1 for (4.3):

$$T^1(g) = \int_W \vec{w} C(g) \exp \{-A(g) [\vec{w} - \vec{c}(g)] \cdot [\vec{w} - \vec{c}(g)]\} d^3 \vec{w}. \quad (4.23)$$

We change the variables in the considered integral according to the relations

$$\vec{w} = \vec{c}(g) + \vec{u}, \quad d^3 \vec{w} = d^3 \vec{u}$$

and the result is

$$\begin{aligned} T^1(g) &= \int_S [\vec{c}(g) + \vec{u}] C(g) \exp [-A(g) \vec{u} \cdot \vec{u}] d^3 \vec{u} = \\ &= C(g) \int_S \vec{c}(g) \exp [-A(g) \vec{u} \cdot \vec{u}] d^3 \vec{u} + C(g) \int_S \vec{u} \exp [-A(g) \vec{u} \cdot \vec{u}] d^3 \vec{u}. \end{aligned} \quad (4.24)$$

From symmetry properties it follows that

$$\int_S \vec{u} \exp[-A(g)\vec{u} \cdot \vec{u}] d^3\vec{u} = 0,$$

therefore

$$\begin{aligned} T^1(g) &= C(g) \int_S \vec{c}(g) \exp[-A(g)\vec{u} \cdot \vec{u}] d^3\vec{u} = \\ C(g)\vec{c}(g) \int_S \exp[-A(g)\vec{u} \cdot \vec{u}] d^3\vec{u} &= n_0(g)\vec{c}(g). \end{aligned} \quad (4.25)$$

For locally equilibrium distribution function $f_0(g, w)$ the moment expression for $T^0(g)$ is $n_0(g)$ (compare (4.10)) and the meaning of (4.25) is that – for locally equilibrium distribution functions – $T^1(g)$ is a product of $T^0(g)$ and the macroscopic four-velocity. It can be checked that this relation is of a general nature. Let us check that the quotient of $T^1(g)$ and $T^0(g)$ is a well-defined four-velocity field on G . According to the definition (1.3), we have to compute the corresponding contraction with a chronological form and check that the result is equal to one

$$\begin{aligned} \left\langle \Psi, \frac{\int_W \vec{w} f(g, \vec{w}) d^3\vec{w}}{\int_W f(g, \vec{w}) d^3\vec{w}} \right\rangle &= \frac{1}{\int_W f(g, \vec{w}) d^3\vec{w}} \left\langle \Psi, \int_W \vec{w} f(g, \vec{w}) d^3\vec{w} \right\rangle = \\ \frac{1}{\int_W f(g, \vec{w}) d^3\vec{w}} \int_W \langle \Psi, \vec{w} f(g, \vec{w}) \rangle d^3\vec{w} &= \\ \frac{1}{\int_W f(g, \vec{w}) d^3\vec{w}} \int_W \langle \Psi, \vec{w} \rangle f(g, \vec{w}) d^3\vec{w} &= \\ \frac{1}{\int_W f(g, \vec{w}) d^3\vec{w}} \int_W f(g, \vec{w}) d^3\vec{w} &= 1. \end{aligned} \quad (4.26)$$

It can be checked that our approach can be generalized to the case of mixtures and gives the proper definition of barycentric (four) velocity [31, 32, 36, 39].

For locally equilibrium states, a notion of temperature is defined in the terms of the moments computed from the locally equilibrium distribution function (4.9) according to the relations (4.17)–(4.19). In general, the notion of temperature outside the local equilibrium is not clear. However, in kinetic theory it is possible to define the “effective temperature” for arbitrary states in the following manner: one defines mean energy of a molecule as

$$\epsilon(g) = \frac{\int_W \frac{1}{2} m [\vec{w} - \vec{c}(g)] \cdot [\vec{w} - \vec{c}(g)] f(g, \vec{w}) d^3\vec{w}}{\int_W f(g, \vec{w}) d^3\vec{w}}, \quad (4.27)$$

and the “effective temperature” for monoatomic gas (without rotational and vibrational degrees of freedom) is defined from (4.27) by the rule

$$T(g) = \frac{2\epsilon(g)}{3k_B}. \quad (4.28)$$

Obviously, the general definition of the mass density is

$$\rho(g) = m \int_W f(g, w) d^3w. \quad (4.29)$$

A locally equilibrium Fermi–Dirac distribution function [19] (in the spinless approximation and formulated invariantly) is defined as

$$f_0(g, w) = \frac{1}{C(g) \exp \left\{ -A(g) \left[\vec{w} - \vec{c}(g) \right] \cdot \left[\vec{w} - \vec{c}(g) \right] \right\} + 1}, \quad (4.30)$$

and locally equilibrium Bose-Einstein distribution function [19] (in the spinless approximation) is defined as

$$f_0(g, w) = \frac{1}{C(g) \exp \left\{ -A(g) \left[\vec{w} - \vec{c}(g) \right] \cdot \left[\vec{w} - \vec{c}(g) \right] \right\} - 1}. \quad (4.31)$$

“Quantum” gases can be discussed in a similar manner but that is outside the scope of the present text.

Chapter 5

Boltzmann equation

Obviously, moment identities for Boltzmann equation are derived from Boltzmann equation, and the standard form of moment identities has been derived from the standard form of Boltzmann equation [56]. In order to write the moment identities for “invariant” moments (defined as integrals computed from the tensor powers of non-relativistic four-velocity, see (4.20)) one needs an invariant form of the Boltzmann equation. However, a precise discussion of the non-relativistic Boltzmann equation requires a discussion of the product of the affine spaces first and some remarks on that subject are given in Appendix E. In this Chapter we shall use a simplified definition of the Boltzmann equation.

It can be easily seen that $G \times W$ has a canonical structure of the seven-dimensional affine space with $T_G \oplus S$ as a translation space. The elements of this seven-dimensional space can be represented as the corresponding pairs $(g, w) \in G \times W$. The operation of “subtracting of elements” of the affine space $G \times W$ can be defined “naturally”, that is for $(g, w) \in G \times W$ and $(g', w') \in G \times W$ one can define the following rule

$$(g, w) - (g', w') = \left(\overrightarrow{g - g'}, \overrightarrow{w - w'} \right) \in T_G \times S, \quad (5.1)$$

where T_G is a translation space of G and S is a translation space of W . In turn, for any finite-dimensional affine space there exists a canonical complete derivative and precise definition of the invariant definition of the Boltzmann equation is related to that “seven-dimensional” complete derivative (see Appendix E).

However, our purpose here is to derive the moment equations for “invariant” moments (4.20) and for that it is sufficient to use the Boltzmann equation written in the form

$$w \odot Df(g, w) + \frac{\vec{f}}{m} \odot \nabla f(g, w) = J(f), \quad (5.2)$$

where \odot means a contraction, $J(f)$ denotes the collision operator, “ D ” in (5.2) can be interpreted as a complete derivative on G and “ ∇ ” is the “three-dimensional” complete derivative on W . In the previous chapter the “invariant” moments have been defined according to

$$T^n(g) = \int_W \otimes^n \vec{w} f(g, \vec{w}) d^3 \vec{w} \quad (5.3)$$

where $d^3 \vec{w}$ is an invariant measure on W , $\otimes^n \vec{w}$ is the “ n -th” tensor power of the non-relativistic four-velocity \vec{w} . One can easily see that the moment equations for the moments of this type can be derived directly from the Boltzmann equation (5.2), after multiplying both sides of (5.2) with $\otimes^n \vec{w}$

$$\otimes^n \vec{w} [w \odot Df(g, w)] + \otimes^n \vec{w} \left[\frac{\vec{f}}{m} \odot \nabla f(g, w) \right] = \otimes^n \vec{w} J(f) \quad (5.4)$$

and integrating both sides of (5.4) with respect to the invariant measure $d^3 \vec{w}$ on W

$$\int_W \left\{ \otimes^n \vec{w} [w \odot Df(g, w)] + \otimes^n \vec{w} \left[\frac{\vec{f}}{m} \odot \nabla f(g, w) \right] \right\} d^3 \vec{w} = \int_W \otimes^n \vec{w} J(f) d^3 \vec{w}. \quad (5.5)$$

From (5.5) one can see that on account of the linearity of the integral (5.5) is equivalent to

$$\int_W \otimes^n \vec{w} [w \odot Df(g, w)] d^3 \vec{w} + \int_W \otimes^n \vec{w} \left[\frac{\vec{f}}{m} \odot \nabla f(g, w) \right] d^3 \vec{w} = \int_W \otimes^n \vec{w} J(f) d^3 \vec{w}. \quad (5.6)$$

In textbooks on kinetic theory (see, for example, [36]) the second term on the l.h.s. of (5.6) is usually transformed to the form of a surface integral and the assumptions on the limit behavior of f are taken. Here we do not want to enter into the problem of formulating Stokes theorem for W space. We are interested in

the properties of the first term on the l.h.s. of (5.6) and (5.6) can be transformed to a form

$$\begin{aligned} & \text{Div} \int_W \otimes^{n+1} \vec{w} f(g, w) d^3 \vec{w} + \\ & \int_W \otimes^n \vec{w} \left[\frac{\vec{f}}{m} \odot \nabla f(g, w) \right] d^3 \vec{w} = \int_W \otimes^n \vec{w} J(f) d^3 \vec{w} \end{aligned} \quad (5.7)$$

where “*Div*” is a four-dimensional “affine” divergence applied already in Chapter three and discussed additionally in Appendix B (see also Appendix E). Invariant moments of the form

$$\int_W \otimes^{n+1} \vec{w} f(g, w) d^3 \vec{w}$$

can be decomposed in a given inertial coordinate system and (after some algebra) transformed into “standard” moment identities known, for example, from [56]. Here we restrict our discussion to a very simple example of the second “invariant” moment $T^{(2)}(g)$ (see (4.20)) and in order to understand what information is contained in $T^{(2)}(g)$ one should remember that on account of (4.21) $T^{(2)}(g)$ determines $T^{(1)}(g)$ according to the relation $T^{(1)}(g) = \Psi \odot T^{(2)}(g)$. In turn, $T^{(1)}(g)$ determines $T^{(0)}(g)$ according to the relation $T^{(0)}(g) = \Psi \odot T^{(1)}(g)$. The explicit integral expression for $T^{(2)}(g)$ is

$$T^{(2)}(g) = \int_W \vec{w} \otimes \vec{w} f(g, w) d^3 \vec{w} \quad (5.8)$$

and one can insert (4.7) in order to change variables in the integral (5.8); from

$$\vec{w} = \vec{w}_I + \vec{u}_I \quad (5.9)$$

one arrives at

$$d^3 \vec{w} = d^3 \vec{u}_I$$

and finally

$$\int_W \vec{w} \otimes \vec{w} f(g, w) d^3 \vec{w} = \int_S [\vec{w}_I + \vec{u}_I] \otimes [\vec{w}_I + \vec{u}_I] f(g, \vec{w}_I + \vec{u}_I) d^3 \vec{u}_I. \quad (5.10)$$

Let us denote

$$f(g, \vec{w}_I + \vec{u}_I) = f_{\vec{w}_I}(g, \vec{u}_I) \quad (5.11)$$

and the expression

$$T^{(2)}(g) = \int_S [\vec{w}_I + \vec{u}_I] \otimes [\vec{w}_I + \vec{u}_I] f_{\vec{w}_I}(g, \vec{u}_I) d^3 \vec{u}_I \quad (5.12)$$

on account of the linearity of the integral can be transformed to give

$$\begin{aligned}
T^{(2)}(g) = & \int_S \vec{w}_I \otimes \vec{w}_I f_{\vec{w}_I}(g, \vec{u}_I) d^3 \vec{u}_I + \int_S \vec{w}_I \otimes \vec{u}_I f_{\vec{w}_I}(g, \vec{u}_I) d^3 \vec{u}_I + \\
& \int_S \vec{u}_I \otimes \vec{w}_I f_{\vec{w}_I}(g, \vec{u}_I) d^3 \vec{u}_I + \int_S \vec{u}_I \otimes \vec{u}_I f_{\vec{w}_I}(g, \vec{u}_I) d^3 \vec{u}_I \quad (5.13)
\end{aligned}$$

In (5.12) \vec{w}_I is a fixed ‘‘affine’’ vector (from W) and it does not belong to the ‘‘domain of integration’’ (over S). Therefore the identity (5.13) can be equivalently written as

$$\begin{aligned}
T^{(2)}(g) = & \vec{w}_I \otimes \vec{w}_I \int_S f_{\vec{w}_I}(g, \vec{u}_I) d^3 \vec{u}_I + \vec{w}_I \otimes \int_S \vec{u}_I f_{\vec{w}_I}(g, \vec{u}_I) d^3 \vec{u}_I + \\
& \left[\int_S \vec{u}_I f_{\vec{w}_I}(g, \vec{u}_I) d^3 \vec{u}_I \right] \otimes \vec{w}_I + \int_S \vec{u}_I \otimes \vec{u}_I f_{\vec{w}_I}(g, \vec{u}_I) d^3 \vec{u}_I. \quad (5.14)
\end{aligned}$$

From the identity (5.14) one can see that $T^{(2)}(g)$ contain information about three ‘‘euclidian’’ quantities: $\int_S f_{\vec{w}_I}(g, \vec{u}_I) d^3 \vec{u}_I$, $\int_S \vec{u}_I f_{\vec{w}_I}(g, \vec{u}_I) d^3 \vec{u}_I$ and $\int_S \vec{u}_I \otimes \vec{u}_I f_{\vec{w}_I}(g, \vec{u}_I) d^3 \vec{u}_I$, being the ‘‘euclidian scalar’’, the ‘‘euclidian vector’’ and the ‘‘euclidian symmetric tensor’’, correspondingly. Such decomposition can be done for a fixed four-velocity \vec{w}_I and depends on the choice of this four-velocity. All these ‘‘euclidean’’ quantities are joined to form one ‘‘invariant’’ moment $T^{(2)}(g)$. Among the components of the ‘‘invariant moment’’ $T^{(2)}(g)$ one can choose sets transforming under the change of the four-velocity \vec{w}_I ‘‘into itself’’ and it can be easily checked that this property has the following set of quantities: $\int_S f_{\vec{w}_I}(g, \vec{u}_I) d^3 \vec{u}_I$, $\int_S \vec{u}_I f_{\vec{w}_I}(g, \vec{u}_I) d^3 \vec{u}_I$ and $\int_S \vec{u}_I \cdot \vec{u}_I f_{\vec{w}_I}(g, \vec{u}_I) d^3 \vec{u}_I$ (the difference with the full set of components of $T^{(2)}(g)$ is that now one takes the integral of the scalar product of velocities $\int_S \vec{u}_I \cdot \vec{u}_I f_{\vec{w}_I}(g, \vec{u}_I) d^3 \vec{u}_I$ instead of the integral of the tensor product of velocities $\int_S \vec{u}_I \otimes \vec{u}_I f_{\vec{w}_I}(g, \vec{u}_I) d^3 \vec{u}_I$). Unfortunately, in this text we have no place for a discussion of the relations between ‘‘invariants’’ and geometrical objects on the Galilean group but we hope to finish that discussion later.

Now we shall show that (5.2) can be transformed to the ‘‘standard’’ non-relativistic Boltzmann kinetic equation. As it has been already mentioned in Introduction (formula (1.30)), the domain of the non-relativistic distribution function is the set $G \times W$ and it can be parameterized according to the formula

$$(t, x^i, u^j) \rightarrow (g_0 + t\vec{w}_I + x^i \vec{e}_i, \vec{w}_I + u^j \vec{e}_j) \in G \times W, \quad (5.15)$$

where $g_0 \in G$ is a fixed space-time point, \vec{w}_I is an arbitrary non-relativistic four-velocity (that is, $\vec{w}_I \in W$) and \vec{e}_j , $j = 1, 2, 3$ is an arbitrary orthonormal reper in S . The inertial coordinate system (5.15) corresponds to the observations of the inertial observer with a world-line

$$R \ni t \rightarrow g_0 + t\vec{w}_I \in H_{[g_0]}. \quad (5.16)$$

The inertial observer with a world-line (5.16) parameterizes the space-time according to the formula

$$R^4 \ni (t, x^1, x^2, x^3) \rightarrow g_0 + t\vec{w}_I + x^i \vec{e}_i \in H_t \subset G, \quad (5.17)$$

while for this inertial observer the “space of the four-velocities” can be parameterized according to

$$R^3 \ni (u^1, u^2, u^3) \rightarrow \vec{w} = \overrightarrow{w(u^j)} = \vec{w}_I + u^j \vec{e}_j \in W. \quad (5.18)$$

The symbol “ D ” in (5.2) means the four-dimensional complete differential (taken with respect to the variables from G) and therefore the “space-time” differential $Df(g, \vec{w})$ is

$$\begin{aligned} Df(g, \vec{w}) &= Df(g_0 + t\vec{w}_I + x^i \vec{e}_i, \vec{w}) = \frac{\partial f(g_0 + t\vec{w}_I + x^i \vec{e}_i, \vec{w})}{\partial t} Dt + \\ &+ \frac{\partial f(g_0 + t\vec{w}_I + x^i \vec{e}_i, \vec{w})}{\partial x^l} Dx^l. \end{aligned} \quad (5.19)$$

In (5.2), a complete derivative $Df(g, \vec{w})$ is contracted with the non-relativistic four velocity \vec{w} and the explicit form of that contraction can be computed after taking into account (5.18), (5.19) and the “duality conditions” (3.8)–(3.11):

$$\begin{aligned} \vec{w} \odot Df(g, w) &= [\vec{w}_I + u^j \vec{e}_j] \odot Df(g_0 + t\vec{w}_I + x^i \vec{e}_i, w) = \\ &[\vec{w}_I + u^j \vec{e}_j] \odot \frac{\partial f(g_0 + t\vec{w}_I + x^i \vec{e}_i, w)}{\partial t} Dt + \\ &[\vec{w}_I + u^j \vec{e}_j] \odot \frac{\partial f(g_0 + t\vec{w}_I + x^i \vec{e}_i, w)}{\partial x^l} Dx^l = \\ &\frac{\partial f(g_0 + t\vec{w}_I + x^i \vec{e}_i, w)}{\partial t} \langle Dt, \vec{w}_I \rangle + \frac{\partial f(g_0 + t\vec{w}_I + x^i \vec{e}_i, w)}{\partial x^l} u^j \langle Dx^l, \vec{e}_j \rangle = \\ &\frac{\partial f(g_0 + t\vec{w}_I + x^i \vec{e}_i, w)}{\partial t} + \frac{\partial f(g_0 + t\vec{w}_I + x^i \vec{e}_i, w)}{\partial x^l} u^j \delta_{lj} = \\ &\frac{\partial f(g_0 + t\vec{w}_I + x^i \vec{e}_i, w)}{\partial t} + u^j \frac{\partial f(g_0 + t\vec{w}_I + x^i \vec{e}_i, w)}{\partial x^j}. \end{aligned} \quad (5.20)$$

The result (5.20) is consistent with the standard form of the Boltzmann equation [19, 36]. In turn, it can be easily checked that the integration in the collision operator can be written in the terms of the euclidian measure on the “four-velocity space”. As it has been already mentioned, our discussion is not complete because we do not discuss the transformations of the second term on the l.h.s. of (5.6), (5.7); in invariant formulation, it requires formulation of Stokes theorem for W space and that problem is outside the scope of the present text.

In (4.26) it has been shown that the quotient of $T^1(g)$ and $T^0(g)$ is a well-defined four velocity field and it is easy to check that this four-velocity field satisfies the following condition:

$$\int_W [\vec{w} - \overrightarrow{c(g)}] f(g, \vec{w}) d^3 \vec{w} = 0. \quad (5.21)$$

Obviously, (5.21) is equivalent to

$$\overrightarrow{c(g)} \int_W f(g, \vec{w}) d^3 \vec{w} = \int_W \vec{w} f(g, \vec{w}) d^3 \vec{w}, \quad (5.22)$$

and, under condition

$$\int_W f(g, \vec{w}) d^3 \vec{w} \neq 0, \quad (5.23)$$

implies that

$$\overrightarrow{c(g)} = \frac{\int_W \vec{w} f(g, \vec{w}) d^3 \vec{w}}{\int_W f(g, \vec{w}) d^3 \vec{w}}. \quad (5.24)$$

In order to check that the definition (5.24) is equivalent to the standard definition of the macroscopic velocity of the monoatomic gas, we have to decompose “macroscopic four-velocity” $\overrightarrow{c(g)}$ and “molecular four-velocity” \vec{w} with respect to the “four-velocity of an inertial observer” \vec{w}_I ; these decompositions are of the form

$$\overrightarrow{c(g)} = [\overrightarrow{c(g)} - \vec{w}_I] + \vec{w}_I = \overrightarrow{c_I(g)} + \vec{w}_I \quad (5.25)$$

and

$$\vec{w} = [\vec{w} - \vec{w}_I] + \vec{w}_I = \vec{u}_I + \vec{w}_I \quad (5.26)$$

and have been already used in Chapter four, (4.5)-(4.7). After inserting (5.25), (5.26) into (5.24) one arrives at:

$$\begin{aligned} \overrightarrow{c}(g) &= \overrightarrow{c_I}(g) + \overrightarrow{w_I} = \frac{\int_W \overrightarrow{w} f(g, \overrightarrow{w}) d^3 \overrightarrow{w}}{\int_W f(g, \overrightarrow{w}) d^3 \overrightarrow{w}} = \frac{\int_W [\overrightarrow{u_I} + \overrightarrow{w_I}] f(g, \overrightarrow{w}) d^3 \overrightarrow{w}}{\int_W f(g, \overrightarrow{w}) d^3 \overrightarrow{w}} = \\ &= \frac{\int_W \overrightarrow{u_I} f(g, \overrightarrow{w}) d^3 \overrightarrow{w}}{\int_W f(g, \overrightarrow{w}) d^3 \overrightarrow{w}} + \frac{\overrightarrow{w_I} \int_W f(g, \overrightarrow{w}) d^3 \overrightarrow{w}}{\int_W f(g, \overrightarrow{w}) d^3 \overrightarrow{w}} = \frac{\int_W \overrightarrow{u_I} f(g, \overrightarrow{w}) d^3 \overrightarrow{w}}{\int_W f(g, \overrightarrow{w}) d^3 \overrightarrow{w}} + \overrightarrow{w_I}. \end{aligned} \quad (5.27)$$

After subtracting $\overrightarrow{w_I}$ from both sides of (5.27) one can see that a standard definition of the macroscopic velocity of monoatomic gas is satisfied:

$$\overrightarrow{c_I}(g) = \frac{\int_W \overrightarrow{u_I} f(g, \overrightarrow{w}) d^3 \overrightarrow{w}}{\int_W f(g, \overrightarrow{w}) d^3 \overrightarrow{w}}. \quad (5.28)$$

Other important definitions from the kinetic theory of monoatomic gases are the density of the molecules

$$n(g) = \int_W f(g, \overrightarrow{w}) d^3 \overrightarrow{w} \quad (5.29)$$

and the mass density

$$\rho(g) = m \int_W f(g, \overrightarrow{w}) d^3 \overrightarrow{w}. \quad (5.30)$$

Our discussion of the Boltzmann equation is uncomplete for many reasons.

We do not discuss the collision term (it can be shown that also the collision term can be written invariantly but this problem shall be not discussed in the present text).

Analysis of Liouvillean part of a Boltzmann equation is not complete; as it has been already mentioned in Introduction, the domain of the non-relativistic distribution function for a monoatomic ideal gas is a product $G \times W$ and the corresponding inertial coordinate systems on the domain of non-relativistic distribution function are defined in (1.30). In this Chapter we use the invariant form of Boltzmann equation given as (5.2) and it is written in terms of a pair of operators “ D ” and “ ∇ ”; it should be stressed that a formal definition of a Liouvillean part of the kinetic equation should be expressed in terms of the complete derivative on $G \times W$ (such an operator exists on $G \times W$ because $G \times W$ posses a canonical structure of a seven-dimensional affine space). Some remarks on that subject are given in Appendix E.

For the Boltzmann equation of the form (5.2) it is possible to derive the invariant form of moment identities. A general form of moment identities for

Boltzmann equation has been published by Banach and Piekarski in 1989 in a standard notation [56]. A transition from one notation to another requires a decomposition of invariant moments into a linear combinations of “standard” moments and, for illustration, such decomposition is done for simple cases.

We do not write anything about kinetic models for mixtures; kinetic models of mixtures can describe also the chemical reactions (see [31, 32, 39]). The standard question of kinetic models concerns the existence of hydrodynamic approximations. Even if hydrodynamic approximations exist, their properties should be consistent with the results of macroscopic description of the reaction-diffusion processes (compare, for example, [30, 39]) and the important question concerns the number of “velocities” in hydrodynamic approximation. It seems that a discussion of that subject would require simultaneous use of Chapman-Enskog method and spectral methods in kinetic theory (see [33], compare also [31, 32, 36]). More remarks on relations between kinetic and phenomenological models of reaction-diffusion processes are given in the next Chapter.

Chapter 6

Final remarks and conclusions

From mathematical point of view, the general scheme applied in this text is that proposed by Sławianowski [1]; one starts with different “models” and modifications of the Galilean space-time and canonical differential operators on those models are investigated. Such an approach is easier than a direct discussion of the Galilean space-time with all its structures. Different differential operators can be used; they can be related with the analysis on manifolds and this is the case of the “dual” approach of Peradzyński (see [4–6]). The other possibility is to apply the analysis on the normed affine spaces as described in the Schwartz’s monograph [11].

The important notion in fluid mechanics is the substantial derivative and its invariant definition has been given in [3]. With its help, the Navier–Stokes–Fourier equations are written invariantly (see Chapter three; (3.40), (3.41) and (3.42)). In turn, the Gibbs identity can be written in terms of the complete derivatives on the Galilean space-time (see (3.49)–(3.53)).

Readers uninterested in Galilean invariance can read the second Chapter only, where the results on acoustics and thermostatics of fluids are written in the standard notation.

It seems that the results concerning the dense fluids are of particular interest. In general, the dense fluids can be defined in many ways:

- one possibility is to define “dense fluids” as fluid systems with local correlations,
- the alternative manner is to define “dense fluids” by the property that its molecules are of a finite dimensions (the well-known example here is the Van der Waals equation),
- one can apply different versions of the kinetic Enskog equation for fluids, maybe some other definitions are also possible.

But here the simplest possible definition is applied, namely the dense fluids are defined by the property that the energy density per unit mass expressed as a function of the temperature and the mass density is a function not only of temperature but also of the mass density. That definition is discussed in detail in [17]. In Chapter two some models of fluids (including dense fluids) are discussed together with sound speeds. It seems that these results can potentially be applied to ultrasonic experiments (in particular, for fluids with the properties radically different than ideal gases; compare [47]). It is worth to mention that the acoustical properties of biological tissues are usually approximated by fluid models (see [47]; chapter “Biomedical Applications”, and [67]).

The important part of non-relativistic physics is related to different aspects of Boltzmann kinetic equation and therefore it seems important to show that also that equation can be written invariantly. Some notions from the kinetic theory (including inertial coordinates for kinetic theory) are mentioned in Introduction. Other expressions (including locally equilibrium distribution functions) are discussed in Chapter four. From the Boltzmann equation it is possible to derive the corresponding moment identities and the general form of moment identities (in standard notation) has been derived by Banach and Piekarski in 1989 [56]. However, one can define also the “invariant” moments and the derivation of the “invariant” moment identities is discussed in Chapter five. In principle, Boltzmann equation is a single-particle Liouville equation with a source term and therefore in order to write Boltzmann equation invariantly one has to give an invariant definition of the Liouville equation first. A rigorous discussion of the invariant form of the Liouville equations is complicated and some its aspects are discussed in Appendix E (however, problems related to symplectic geometry are omitted).

The above discussed items concerned Boltzmann equation for monoatomic gas but kinetic models are applied also in mixtures and modelling reaction - diffusion processes [31, 32, 36] and it would be interesting to use our approach for them.

Other problems concern diffusion in fluids. In particular, till now the model of diffusion described in [50–54] has not been adopted to fluids and nobody knows how such an adaptation would look like. Also the consistency conditions between corresponding kinetic and phenomenological models of diffusion in fluids are not known.

Some mathematical aspects of our analysis are also interesting.

The most important fact is that Sławianowski’s “models” can be described in terms of the “ Γ -structures” (in the sense of Rychlewski [45]) and we hope to discuss that in detail later.

It is also worth to compare invariants discussed in the present text with “relativistic” invariants [58–65] and the immediate observation is that in both cases the theory of representation of the groups has not been used.

As it has been already mentioned, for our “invariant” equations one can use different differential operators and in our considerations two options have been used: the “dual” approach of Peradzyński ([4–6]) and the analysis on the normed affine spaces described by Schwartz ([11]). In particular, the invariant definition of symmetric conservative and symmetric hyperbolic systems [7] and invariant analysis of Boillat’s approach [8] have been done according to the Peradzyński’s approach and it seems interesting to discuss their reformulation in Schwartz’s language. Also the Navier–Stokes–Fourier equations can be written invariantly also in the formalism introduced in [4] but here we have no place for that and we hope to make a detailed discussion of that subject later.

In this paper we do not discuss the full group of automorphisms of the Galilean space-time but we restrict our attention to such automorphisms that could impose similar restrictions to those corresponding to the “principle of material indifference”. According to Sławianowski’s scheme, the simplest possible “model” of Galilean space-time is the four-dimensional affine space and then one can distinguish subgroups of automorphisms with the affine straight lines as the set of fixed points.

For the “amorphous Galilean space-time” the corresponding automorphisms are shortly described at the end of Appendix C.

For the “Galilean space-time with measurable time distances” those automorphisms are mentioned at the end of Appendix D.

For Galilean space-time they are discussed explicitly in Appendix F.

It is worth to remember that some opinions on “material indifference” are related with the kinetic theory; according to Wilmański [13], a vehement discussion of the status of the material objectivity principle started in 1966 with the PhD Thesis of Ingo Muller, who showed that the kinetic theory of gases leads to the non-objective macroscopical constitutive laws for such quantities as the heat flux. One can easily checked that analogous automorphisms can be defined for the “phase space” of the Boltzmann equation. Moreover, also for that case it is possible to defined simplified “models” of the phase space.

In our opinion, the main ideas of the present text can be concluded as follows:

- from the point of view of applications; the most important part of this text is Chapter two, its results could help to formulate the acoustics of dense fluids more precisely. We hope to obtain more results later.
- from the “theoretical” point of view; it has been shown that the Navier–Stokes–Fourier equations can be written invariantly and some invariant

aspects of the non-relativistic kinetic theory have been also discussed. It could be interesting to determine other non-relativistic theories that can be written in a similar manner. More results on Galilean invariance shall be given in [87].

Chapter 7

Summary

The differential operators on normed affine spaces are described in Schwartz's monograph [11] and the canonical differential operators on Minkowski space-time and Galilean space-time are their particular cases. The general scheme applied in this text is that proposed by Ślawianowski [1]; one starts with different "models" of Galilean space-time and canonical differential operators on those models are described in Appendices.

It seems that those models can be written as the Γ -structures in a sense of Rychlewski [45] but this aspect is outside the scope of the present text.

The alternative approach to differential operators on affine spaces (as well as affine spaces with additional structures) can be formulated by means of the "dual" approach of Peradzyński [4–10].

The "substantial derivative" of fluid mechanics has been identified with a directional derivative in a direction of the non-relativistic four-velocity [2, 3, 11]; therefore the Navier–Stokes–Fourier equations can be written invariantly. The invariant interpretation of the Gibbs identity is given (see Eqs. (3.43)–(3.53)). Some invariant aspects of the non-relativistic kinetic theory are also discussed, together with problems concerning the Galilean symmetry.

The invariant formulation of the Navier–Stokes–Fourier equations simplifies a discussion of "thermodynamics" and the results are described in the second chapter using the standard notation, for the convenience of the readers uninterested in Galilean invariance. The fluid is described in terms of the energy density $E(\rho, T)$ (per unit mass) as a function of the mass density ρ and the temperature T and the pressure $p(\rho, T)$ as a function of ρ and T . According to [16, 17], the case $E = E(T)$ can be called "a generalized ideal gas" while by "a dense fluid" one can mean medium with the energy density (per unit mass) depending not only on the temperature but also on the mass density. New solutions of Gibbs identity

have been obtained; some of them describe dense fluids and the corresponding sound speeds are computed for simple models. It can be observed that from our approach one can derive virial expansions. It is hoped that the proposed approach can have many different applications; for example, it can be applied in medical acoustics where biological tissues are often modelled as the dense fluids [67].

Appendix A

Finite dimensional affine spaces

In this **Appendix**, the most important facts concerning the affine spaces are presented. Only real affine spaces are considered. The definition of an affine space is taken from Sławianowski's monograph (see [1], p. 78) and Komorowski's textbook (see [20], p. 247):

Definition 1. An affine space is a triple $(A, T_A, -)$ where A is an underlying set (point-field, in the terminology of Weyl), T_A is vector space, and “ $-$ ” assigns vectors to pairs of points

$$A \times A \ni (a, b) \rightarrow \overrightarrow{a - b} \in T_A \quad (\text{A.1})$$

in such a way that

I.
$$\overrightarrow{a - b} + \overrightarrow{b - c} + \overrightarrow{c - a} = 0 \quad \text{for } a, b, c \in A; \quad (\text{A.2})$$

II. the mapping $A \ni a \rightarrow \overrightarrow{a - b} \in T_A, \quad b \in A,$ is a bijection. (A.3)

If T_A has a dimension n , then $(A, T_A, -)$ is called the n -dimensional affine space. Sometimes the underlying set A itself is called the affine space. The vector space T_A is called a space tangent to A (sometimes a translation space of A), and its elements are called the translation vectors. The most important properties and definitions are

Observation 1.

$$\overrightarrow{a - a} = 0. \quad (\text{A.4})$$

$$\overrightarrow{a - b} = -\overrightarrow{b - a}. \quad (\text{A.5})$$

Definition 2. The vectors from T_A can be added to the points from A according to the following rule

$$a + \vec{\alpha} = b \iff \overrightarrow{a - b} = \vec{\alpha}, \quad (\text{A.6})$$

and the rule for subtraction is

$$a - \vec{\alpha} = a + [-\vec{\alpha}]. \quad (\text{A.7})$$

Observation 2.

$$a + \overrightarrow{b - a} = b, \quad (\text{A.8})$$

$$(a + \vec{\alpha}) + \vec{\beta} = \vec{\alpha} + (a + \vec{\beta}), \quad (\text{A.9})$$

$$(a + \vec{\alpha}) - b = \vec{\alpha} + \overrightarrow{a - b} = a - (b - \vec{\alpha}). \quad (\text{A.10})$$

In principle, in (A.10) one should use $\overrightarrow{(a + \vec{\alpha}) - b}$ instead of $(a + \vec{\alpha}) - b$ but simplified notation seems to be more convenient here.

Definition 3. An **Euclidean space** is such an affine space $(E, T_E, -)$, that T_E is the unitary space (with a fixed scalar product).

Definition 4. An **Automorphism of an affine space** $(A, T_A, -)$ is such a mapping $f : A \rightarrow A$, that

$$f(a) - f(a') = f_*(a - a'), \quad (\text{A.11})$$

where a and a' are arbitrary elements of A and f_* is a linear bijection from T_A into T_A . This definition is a particular case of a definition of the affine mapping between the affine spaces, given on p. 250 of Komorowski's textbook [20]. Smooth mappings between the finite-dimensional affine spaces can be locally "approximated" by the affine mappings and the possibility of such an approximation means that the considered mapping has the complete derivative. In Schwartz's monograph [11] this definition is discussed in detail and it is formulated for the case of the mappings between the normed affine space. In order to use it for the mappings between the finite-dimensional affine spaces some additional remarks are necessary (see also [1], p. 81).

Let us consider two finite-dimensional affine spaces $(A, T_A, -)$ and $(B, T_B, -)$. Let $f : A \rightarrow B$ be a mapping from A to B :

$$A \ni a \rightarrow f(a) \in B.$$

It is well-known that all norms on a finite-dimensional vector space are equivalent; let the set of all norms on T_A be denoted as $\| \cdot \|_\gamma$, $\gamma \in \Gamma$ and let the set of all norms on T_B be denoted as $\| \cdot \|_\omega$, $\omega \in \Omega$.

Definition 5. We say that f has a complete derivative at $a \in A$ if

$$f(b) - f(a) = f_a \left[\overrightarrow{b - a} \right] + \varepsilon(b, a)$$

and

$$\lim \frac{\|\varepsilon(b, a) \cdot \|\omega\|}{\|\overrightarrow{b-a}\|_\gamma} = 0 \quad \text{for} \quad \|\overrightarrow{b-a}\|_\gamma \rightarrow 0, \quad (\text{A.12})$$

it should be stressed that the condition (A.12) does not depend on the choice of norms in T_A and T_B .

From the definition (A.7) one can see that (A.11) can be equivalently written as

$$f(a) = f(a') + f_*(a - a'). \quad (\text{A.13})$$

The definition of an affine space $\overrightarrow{a-b}$ implies that for a fixed $b \in A$ all elements $a \in A$ can be represented by vectors $\overrightarrow{a-b}$. In particular,

$$f(a) - f(b) = f_*(a - b) \quad (\text{A.14})$$

is equivalent to

$$[f(a) - b] = [f(b) - b] + f_*(a - b). \quad (\text{A.15})$$

The expression (A.15) represents an automorphism of the affine space A as a superposition of the linear mapping (acting on $\overrightarrow{a-b}$) with the corresponding translation. From the rules of affine geometry it follows that (A.15) can be written in the following alternative form:

$$f(a) = b + [f(b) - b] + f_*(a - b) \quad (\text{A.16})$$

(obviously, (A.16) is valid for all possible points a, b of A).

General automorphisms can be further restricted after imposing additional conditions on them. In particular, such conditions can define a set of fixed points for a mapping considered. In general, a is a fixed point of a mapping f if

$$a = f(a). \quad (\text{A.17})$$

For our purposes, a sets of fixed points that form ‘‘affine straight lines’’ are of a particular interest. In a four-dimensional affine space one can introduce parametric description of straight lines

$$R \ni \tau \rightarrow b + \tau \vec{L} = a(\tau) \in A, \quad \vec{L} \in T_A, \vec{L} \neq \vec{0}. \quad (\text{A.18})$$

Of course, such parametric description is not uniquely defined and in order to discuss that effect one can take two different values τ_1, τ_2 of a parameter; they define two points of a considered straight line

$$a(\tau_1) = b + \tau_1 \vec{L} \in A \quad (\text{A.19})$$

and

$$a(\tau_2) = b + \tau_2 \vec{L} \in A. \quad (\text{A.20})$$

According to the rules of affine geometry, the points (A.19) and (A.20) can be subtracted and the result is

$$a(\tau_1) - a(\tau_2) = [b + \tau_1 \vec{L}] - [b + \tau_2 \vec{L}] = \tau_1 \vec{L} - \tau_2 \vec{L} = (\tau_1 - \tau_2) \vec{L} \in T_A. \quad (\text{A.21})$$

After comparing (A.18) and (A.21) one can see that for a given straight line with a parametric description (A.18) there exists a one-dimensional vector space of “directions” and for two different parametric descriptions (A.18) and

$$R \ni \tau' \rightarrow b' + \tau' \vec{L}' = a(\tau') \in A, \quad \vec{L}' \in T_A, \quad \vec{L}' \neq \vec{0} \quad (\text{A.22})$$

the condition that (A.18) and (A.22) parameterize the same set

$$b + \tau \vec{L} = b' + \tau' \vec{L}' \quad (\text{A.23})$$

is that there exists $\alpha \in R$ that satisfies

$$\overrightarrow{b - b'} = \alpha \vec{L}. \quad (\text{A.24})$$

Let the points of the straight line (A.18) be the fixed points of an automorphism (A.16):

$$\tau \in R \rightarrow f(b + \tau \vec{L}) = b + \tau \vec{L}. \quad (\text{A.25})$$

The condition (A.25) is satisfied for all real τ and for $\tau = 0$ it implies that

$$f(b) = b. \quad (\text{A.26})$$

After inserting (A.26) into (A.16) one obtains

$$f(a) = b + f_*(a - b). \quad (\text{A.27})$$

Now, (A.27) is satisfied for arbitrary a and therefore it has to be satisfied also for $a = b + \tau \vec{L}$:

$$[f(b + \tau \vec{L}) - b] = f_*((b + \tau \vec{L}) - b). \quad (\text{A.28})$$

In view of (A.25), (A.28) is equivalent to

$$[(b + \tau \vec{L}) - b] = f_*(\tau \vec{L}) \quad (\text{A.29})$$

From the definition (A.11) one knows that f_* is a linear mapping

$$\tau \vec{L} = f_*(\tau \vec{L}) \quad (\text{A.30})$$

and therefore one arrives at the condition

$$\vec{L} = f_* \left(\vec{L} \right). \quad (\text{A.31})$$

In order to determine the explicit form of f_* implied by the condition (A.31) it is convenient to choose in T_A such a basis that \vec{L} belongs to that basis and the remaining vectors from the basis are $\vec{E}_1, \vec{E}_2, \vec{E}_3$. Then it is possible to introduce on A the following affine coordinate system:

$$(\tau, Z^1, Z^2, Z^3) \rightarrow b + \tau \vec{L} + Z^\beta \vec{E}_\beta = a \left(\tau, Z^\beta \right) \in A. \quad (\text{A.32})$$

The dual basis of the basis $\left\{ \vec{L}, \vec{E}_1, \vec{E}_2, \vec{E}_3 \right\}$ is a set of forms F^0, F^1, F^2, F^3 from the translation space T_A^* that satisfies the following duality conditions:

$$\begin{aligned} \langle F^0, \vec{L} \rangle &= 1, \\ \langle F^0, \vec{E}_\beta \rangle &= 0, \quad \beta = 1, 2, 3. \\ \langle F^\beta, \vec{L} \rangle &= 0, \quad \beta = 1, 2, 3. \\ \langle F^\beta, \vec{E}_\alpha \rangle &= \delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3. \end{aligned} \quad (\text{A.33})$$

According to the isomorphisms between the linear mappings and the tensor spaces, any linear mapping from T_A into T_A can be identified with the corresponding tensor from $T_A \otimes T_A^*$. In particular, the linear mapping f_* standing in the definition of the affine automorphism f can be decomposed in the above written bases:

$$f_* = A_0 \vec{L} \otimes F^0 + A_0^\alpha \vec{E}_\alpha \otimes F^0 + A_\beta \vec{L} \otimes F^\beta + A_\beta^\alpha \vec{E}_\alpha \otimes F^\beta, \quad \alpha, \beta = 1, 2, 3. \quad (\text{A.34})$$

The action of (A.34) on \vec{L} can be computed explicitly, after making use of the duality conditions (A.33):

$$\begin{aligned} f_* \left(\vec{L} \right) &= \left[A_0 \vec{L} \otimes F^0 + A_0^\alpha \vec{E}_\alpha \otimes F^0 + A_\beta \vec{L} \otimes F^\beta + A_\beta^\alpha \vec{E}_\alpha \otimes F^\beta \right] \odot \vec{L} = \\ &= A_0 \vec{L} \langle F^0, \vec{L} \rangle + A_0^\alpha \vec{E}_\alpha \langle F^0, \vec{L} \rangle + A_\beta \vec{L} \langle F^\beta, \vec{L} \rangle + A_\beta^\alpha \vec{E}_\alpha \langle F^\beta, \vec{L} \rangle = \\ &= A_0 \vec{L} + A_0^\alpha \vec{E}_\alpha. \end{aligned} \quad (\text{A.35})$$

Therefore, (A.31) can be satisfied under the condition that

$$A_0 = 1, \quad A_0^\alpha = 0, \quad (\text{A.36})$$

what means that the explicit form of f_* is

$$f_* = \vec{L} \otimes F^0 + A_\beta \vec{L} \otimes F^\beta + A_\beta^\alpha \vec{E}_\alpha \otimes F^\beta, \quad \alpha, \beta = 1, 2, 3. \quad (\text{A.37})$$

Now it is possible to describe explicitly the action of the considered automorphism on the arbitrary point a of the affine space A ; it follows from the general formula (A.27), after taking into account (A.37) and making use of (A.32):

$$\begin{aligned} f(a) &= b + f_*(a - b) = b + f_* \left(\left[b + \tau \vec{L} + Z^\alpha \vec{E}_\alpha \right] - b \right) = b + f_* \left(\tau \vec{L} + Z^\alpha \vec{E}_\alpha \right) = \\ &= b + \left[\vec{L} \otimes F^0 + A_\beta \vec{L} \otimes F^\beta + A_\beta^\gamma \vec{E}_\gamma \otimes F^\beta \right] \odot \left[\tau \vec{L} + Z^\alpha \vec{E}_\alpha \right] = \\ &= b + \left[\vec{L} \otimes F^0 + A_\beta \vec{L} \otimes F^\beta + A_\beta^\gamma \vec{E}_\gamma \otimes F^\beta \right] \odot \left[\tau \vec{L} \right] + \\ &= \left[\vec{L} \otimes F^0 + A_\beta \vec{L} \otimes F^\beta + A_\beta^\gamma \vec{E}_\gamma \otimes F^\beta \right] \odot \left[Z^\alpha \vec{E}_\alpha \right] = \\ &= b + \vec{L} \langle F^0, \tau \vec{L} \rangle + A_\beta \vec{L} \langle F^\beta, \tau \vec{L} \rangle + A_\beta^\gamma \vec{E}_\gamma \langle F^\beta, \tau \vec{L} \rangle + \\ &= \vec{L} \langle F^0, Z^\alpha \vec{E}_\alpha \rangle + A_\beta \vec{L} \langle F^\beta, Z^\alpha \vec{E}_\alpha \rangle + A_\beta^\gamma \vec{E}_\gamma \langle F^\beta, Z^\alpha \vec{E}_\alpha \rangle = \\ &= b + \tau \vec{L} \langle F^0, \vec{L} \rangle + \tau A_\beta \vec{L} \langle F^\beta, \vec{L} \rangle + \tau A_\beta^\gamma \vec{E}_\gamma \langle F^\beta, \vec{L} \rangle + \\ &= Z^\alpha \vec{L} \langle F^0, \vec{E}_\alpha \rangle + Z^\alpha A_\beta \vec{L} \langle F^\beta, \vec{E}_\alpha \rangle + Z^\alpha A_\beta^\gamma \vec{E}_\gamma \langle F^\beta, \vec{E}_\alpha \rangle. \end{aligned} \quad (\text{A.38})$$

After taking into account the duality conditions (A.33) one can determine the explicit form of the action of our automorphism on the arbitrary point $a = b + \tau \vec{L} + Z^\alpha \vec{E}_\alpha$ of A :

$$f(a) = f \left(b + \tau \vec{L} + Z^\alpha \vec{E}_\alpha \right) =$$

$$b + \tau \vec{L} + Z^\alpha A_\beta \vec{L} \langle F^\beta, \vec{E}_\alpha \rangle + Z^\alpha A_\beta^\gamma \vec{E}_\gamma \langle F^\beta, \vec{E}_\alpha \rangle =$$

$$b + \tau \vec{L} + Z^\alpha A_\beta \vec{L} \delta_{\alpha\beta} + Z^\alpha A_\beta^\gamma \vec{E}_\gamma \delta_{\alpha\beta} = b + \tau \vec{L} + Z^\alpha A_\alpha \vec{L} + Z^\alpha A_\alpha^\gamma \vec{E}_\gamma. \quad (\text{A.39})$$

For

$$Z^\alpha = 0, \quad \alpha = 1, 2, 3, \quad (\text{A.40})$$

(A.39) reduces to

$$f \left(b + \tau \vec{L} \right) = b + \tau \vec{L}, \quad (\text{A.41})$$

what is consistent with the assumption (A.20).

The notation applied in (A.39) is not quite precise because the quantity $b + \tau \vec{L}$ has a twofold meaning:

- the first meaning is that the image of the mapping

$$R \ni \tau \rightarrow b + \tau \vec{L}$$

defines the set of fixed points for our automorphism;

- the second meaning is that it is a part of coordinization of the space A .

Therefore it is reasonable to introduce the alternative, more precise notation for (A.39)

$$f(a) = f_{b, \vec{L}} \left(b + \tau \vec{L} + Z^\alpha \vec{E}_\alpha \right) = b + \tau \vec{L} + Z^\alpha A_\alpha \vec{L} + Z^\alpha A_\alpha^\gamma \vec{E}_\gamma. \quad (\text{A.42})$$

We shall see later how the expressions (A.39) and (A.42) are modified for “amorphous Galilean space-time”, “Galilean space-time with measurable time distances” and, finally, in Appendix F, for the physically interesting case of a “real” Galilean space-time.

Appendix B

Differential operators on the finite-dimensional affine space

As it has been already mention in Preface, the differential operators on affine spaces can be introduced in many ways but here we shall use the approach based on the Schwartz's monograph [11] and applied in [3]; a complete derivative on a finite-dimensional affine space can be defined according to general definition of a complete derivative on a normed affine space because all norms on a finite dimensional vector space are equivalent. According to the notation of Appendix A, let T_A denote the translation space of a finite-dimensional affine space and let the set of equivalent norms on T_A be denoted as $\| \cdot \|_\gamma, \gamma \in \Gamma$. One can define a set of equivalent metrics on A in the following manner:

$$A \ni a, a' \rightarrow \rho(a, a') = \| a - a' \|_\gamma, \gamma \in \Gamma. \quad (\text{B.1})$$

All these norms define the same metrizable topology (identical with the “manifold topology”, existing on the four-dimensional affine space as a “finite dimensional manifold”). Therefore, for the mappings between the finite-dimensional affine spaces one can use the notion of a complete derivative (according to Schwartz's theory) and, in our text, this complete derivative is usually denoted as “ D ”.

According to the standard approach of differential geometry, one can define “tangent vectors” and “cotangent forms” to any local coordinate system (see, for example Wintgen and Sulanke's monograph [22]). Moreover, for the affine coordinate systems on a finite-dimensional affine space, it is possible to define also “affine differential quotients”. In particular, for the affine coordinates on A

$$(Z^1, \dots, Z^K) \rightarrow a + Z^\beta \overrightarrow{E}_\beta \in A \quad (\text{B.2})$$

(where $\beta = 1, 2, \dots, K = \dim T_A$, $a \in A$, and \overrightarrow{E}_β is a basis in T_A), one can compute the tangent vector corresponding to the coordinate Z^1 explicitly

$$\overrightarrow{\frac{\partial}{\partial Z^1}} = \lim_{Z^{1'} \rightarrow Z^1} \frac{[Z^{1'} E_1 + Z^{\gamma(1)} E_{\gamma(1)}] - [Z^1 E_1 + Z^{\gamma(1)} E_{\gamma(1)}]}{Z^{1'} - Z^1} = \overrightarrow{E}_1, \quad (\text{B.3})$$

where

$$\gamma(1) = 2, 3, \dots, K = \dim T_A.$$

Similarly, for $\beta \neq 1$ one has

$$\overrightarrow{\frac{\partial}{\partial Z^\beta}} = \overrightarrow{E}_\beta. \quad (\text{B.4})$$

The fields of dual base forms of the coordinate system (B.2) satisfy the defining relations

$$\left\langle DZ^l, \overrightarrow{\frac{\partial}{\partial Z^m}} \right\rangle = \left\langle DZ^l, \overrightarrow{E}_m \right\rangle = \delta_{lm}, \quad l, m = 1, 2, \dots, K = \dim T_A. \quad (\text{B.5})$$

A complete derivative on affine space has been defined in [11] (see also [15] and Appendix A) and the simplest possible definition is to define it in the affine coordinates (B.2).

For the real functions on the finite-dimensional affine space

$$A \ni a \rightarrow \Phi(a) \in R \quad (\text{B.6})$$

the complete derivative can be defined as

$$D\Phi(a) = \frac{\partial \Phi(a)}{\partial Z^\beta} DZ^\beta. \quad (\text{B.7})$$

In turn, let us consider the case of a vector field on the affine space, which is – by definition – the mapping that assigns vectors from T_A to the elements of A

$$A \ni a \rightarrow \overrightarrow{B}(a) \in T_A. \quad (\text{B.8})$$

For the vector fields on A one can use the formalism of the “vector-valued differential forms” but the simplest approach is to write the vector field $\overrightarrow{B}(a)$ in the affine coordinates (B.2):

$$\overrightarrow{B}(a) = B^\beta(a) \overrightarrow{E}_\beta \in T_A. \quad (\text{B.9})$$

A complete derivative of the vector field (B.8)–(B.9) can be defined according to the following rule

$$D\overrightarrow{B}(a) = \overrightarrow{E}_\alpha \otimes DB^\alpha(a) \in T_A \otimes T_A^*, \quad (\text{B.10})$$

where $DB^\alpha(a)$, $\alpha = 1, 2, \dots, K = \dim T_A$ are complete derivatives of the components (every component $B^\alpha(a)$ is a real function on the affine space A). The explicit expression for $DB^\alpha(a)$ is

$$DB^\alpha(a) = \frac{\partial B^\alpha(a)}{\partial Z^\beta} DZ^\beta \quad (\text{B.11})$$

(see [3]) and therefore the final formula for $D\overrightarrow{B(a)}$ is

$$D\overrightarrow{B(a)} = \overrightarrow{E}_\alpha \otimes \frac{\partial B^\alpha(a)}{\partial Z^\beta} DZ^\beta. \quad (\text{B.12})$$

One can see that both indices of (B.12) belong to the “dual” spaces and therefore it is possible to define the corresponding “contraction” (which is defined according to the rules (A.17); for the operation of contraction we use the standard symbol “ Tr ” and it is easy to check that

$$\begin{aligned} Tr \left\{ D\overrightarrow{B(a)} \right\} &= Tr \left\{ \overrightarrow{E}_\alpha \otimes \frac{\partial B^\alpha(a)}{\partial Z^\beta} DZ^\beta \right\} = \frac{\partial B^\alpha(a)}{\partial Z^\beta} Tr \left\{ \overrightarrow{E}_\alpha \otimes DZ^\beta \right\} = \\ &= \frac{\partial B^\alpha(a)}{\partial Z^\beta} \left\langle DZ^\beta, \overrightarrow{E}_\alpha \right\rangle = \frac{\partial B^\alpha(a)}{\partial Z^\beta} \delta_\alpha^\beta = \frac{\partial B^\alpha(a)}{\partial Z^\alpha}. \end{aligned} \quad (\text{B.13})$$

For obvious reasons, (B.13) can be called a “divergence” of a vector field on a finite-dimensional affine space. Quantities of that kind are very important on the Galilean space-time and related spaces. For simplicity, we apply the notation that the “divergence” in (B.13) is denoted as “ Div ”, that is

$$Div \overrightarrow{B(a)} = Tr \left\{ D\overrightarrow{B(a)} \right\} = \frac{\partial B^\alpha(a)}{\partial Z^\alpha}. \quad (\text{B.14})$$

In general, the definition of a “complete derivative” can be formulated naturally for the tensor fields on affine spaces but the definition of a “four-dimensional divergence” of an affine tensor field requires that the considered tensor field has at least one contravariant index.

Now we can better comment our notation from the expression (5.7) because the integral standing under the symbol “ Div ” is defined by the integral

$$\int_W \otimes^{n+1} \overrightarrow{w} f(g, w) d^3 \overrightarrow{w} \quad (\text{B.15})$$

and from the properties of the domain of integration it follows that (B.13) is a totally contravariant tensor field on the Galilean space-time G . The only difference is that in this Appendix we discuss “structureless” affine spaces and on

the Galilean space-time it is convenient to use the inertial atlas. After making explicit computations in the inertial atlas it can be checked that our form of moment equation is consistent with the standard one.

The notion of the directional derivative on the affine space is defined in Schwartz's monograph as the contraction of the complete derivative and the vector field on the affine space. Such definition of the directional derivative can be applied to real functions, vector functions or the tensor fields but for applications in fluid mechanics the directional derivatives of scalar and vector functions are of special importance. Let us compute the directional derivatives of the fields $\Phi(a)$ and $\overrightarrow{B}(a)$ along the vector field $\overrightarrow{H}(a)$

$$A \ni a \rightarrow \overrightarrow{H}(a) = H^\alpha(a) \vec{E}_\alpha \in T_A; \quad (\text{B.16})$$

the corresponding directional derivatives are defined as

$$\begin{aligned} \text{Tr} \left\{ D\overrightarrow{B}(a) \otimes \overrightarrow{H}(a) \right\} &= \text{Tr} \left\{ \left[\vec{E}_\alpha \otimes \frac{\partial B^\alpha(a)}{\partial Z^\beta} DZ^\beta \right] \otimes \left[H^\alpha(a) \vec{E}_\alpha \right] \right\} = \\ H^\alpha(a) \frac{\partial B^\alpha(a)}{\partial Z^\beta} \left\langle DZ^\beta, \vec{E}_\alpha \right\rangle &= H^\alpha(a) \frac{\partial B^\alpha(a)}{\partial Z^\beta} \delta_{\beta\alpha} = H^\alpha(a) \frac{\partial B^\alpha(a)}{\partial Z^\alpha} \end{aligned} \quad (\text{B.17})$$

and

$$\text{Tr} \left\{ D\Phi \otimes \overrightarrow{H}(a) \right\} = H^\alpha(a) \frac{\partial \Phi(a)}{\partial Z^\alpha}. \quad (\text{B.18})$$

In [11], Schwartz denotes (B.17) as

$$\text{Tr} \left\{ D\overrightarrow{B}(a) \otimes \overrightarrow{H}(a) \right\} = D_{\overrightarrow{H}(a)} \overrightarrow{B}(a) \quad (\text{B.19})$$

and (B.18) as

$$\text{Tr} \left\{ D\Phi \otimes \overrightarrow{H}(a) \right\} = D_{\overrightarrow{H}(a)} \Phi. \quad (\text{B.20})$$

We shall alternatively use the notation

$$\text{Tr} \left\{ D\overrightarrow{B}(a) \otimes \overrightarrow{H}(a) \right\} = \overrightarrow{H}(a) \odot D\overrightarrow{B}(a) = H^\alpha(a) \frac{\partial B^\alpha(a)}{\partial Z^\alpha} \quad (\text{B.21})$$

and

$$\text{Tr} \left\{ D\Phi(a) \otimes \overrightarrow{H}(a) \right\} = \overrightarrow{H}(a) \odot D\Phi(a) = H^\alpha(a) \frac{\partial \Phi(a)}{\partial Z^\alpha}. \quad (\text{B.22})$$

Later we shall see that this is related with the “substantial derivative” but this relation cannot be observed on “pure” affine space but it requires more “specific” structure. Namely, it requires “the Galilean space-time with measurable time distances”; obviously, it exists also on the Galilean space-time.

Appendix C

The amorphous Galilean space-time

The simplest possible “model” of the Galilean space-time is the “amorphous Galilean space-time” (defined in [1] on p. 382). Its structure is given as a four-dimension affine space with a distinguished three-dimension vector subspace S in its translation space T_A . This structure can be denoted in the following way:

$$(A, T_A, -, S), \quad (\text{C.1})$$

where the meanings of A, T_A , and “ $-$ ” are similar to that from Appendix A. For (C.1), one can define the following equivalence relation

$$a, a' \in A, \quad a \sim a' \Leftrightarrow \overrightarrow{a - a'} \in S. \quad (\text{C.2})$$

The equivalence classes of the equivalence relation (C.2) are denoted $[a]$ or $A_{[a]}$. It can be easily checked that these equivalence classes are the three-dimensional parallel hyperplanes in A . From the definition given in Appendix A it follows that every hyperplane contained in the affine space is the affine space itself. As it has been mentioned in Appendix B, on every finite-dimensional affine space there exists a “canonical” complete derivative. Therefore, in the “amorphous Galilean space-time” there exist two complete derivatives: one complete derivative is “the four-dimensional one” (and it is denoted as “ D ”, in accordance with the notation of the Appendix B) and the “three-dimensional” complete derivatives existing on the affine spaces $A_{[a]}$. Those “three-dimensional” complete derivatives shall be denoted as “ ∇ ”.

In **Appendix B** the notion of a “divergence” of a vector field on the affine space has been defined. In the case of the “amorphous Galilean space-time” one

has two different complete derivatives “ D ” and “ ∇ ”; therefore one can define more differential operators. For example, for the vector fields on A and with values in T_A

$$A \ni a \rightarrow \overrightarrow{B(a)} \in T_A \quad (\text{C.3})$$

in the amorphous Galilean space-time different quantities can be defined; the first is the “four-dimensional” complete derivative $\overrightarrow{DB(a)}$,

$$A \ni a \rightarrow \overrightarrow{DB(a)} \in T_A \otimes T_A^* \quad (\text{C.4})$$

and the second one is the quantity $\nabla \overrightarrow{B(a)}$,

$$A \ni a \rightarrow \nabla \overrightarrow{B(a)} \in T_A \otimes S^* \quad (\text{C.5})$$

where S^* is the dual space of S .

It is important to mention that for (C.4) it is possible to define the contraction of both indices (because T_A and T_A^* belong to “dual” spaces) but for (C.5) the contraction cannot be defined (because T_A and S^* are not “dual” vector spaces).

Among all tensor fields on the “amorphous Galilean space-time” and with values in tensor products of T_A and T_A^* one can distinguish the tensor fields with values in tensor products of S and S^* . Such tensor fields shall be called “spatial”. In order to discuss its properties in more detail it is useful to introduce the affine atlas “adjusted to the structure of an amorphous Galilean space-time”. Among all bases in T_A one can distinguish such bases that contain three vectors from S and exactly one vector from T_A which does not belong to S .

Such basis can be denoted $\{\vec{L}, \vec{E}_1, \vec{E}_2, \vec{E}_3\}$ where $\vec{L} \in T_A$, $\vec{L} \neq \vec{0}$, $\vec{L} \notin S$ and \vec{E}_α , $\alpha = 1, 2, 3$ is a basis in S .

As it has been already mentioned in Appendix B, the complete derivative “ D ” can be defined for all tensor fields on the finite dimensional affine space A and therefore it can be defined also on “spatial” tensors. However, in order to discuss also other differential operators one has to introduce “the affine atlas adjusted to the structure of the amorphous Galilean space-time” and its coordinate systems are of the form

$$R^4 \ni (\tau, Z^1, Z^2, Z^3) \rightarrow b + \tau \vec{L} + Z^\alpha \vec{E}_\alpha = a(\tau, Z^\alpha) \in A, \quad \alpha = 1, 2, 3. \quad (\text{C.6})$$

That expression is similar to (A.32) but the important difference is that now the vectors $\vec{E}_1, \vec{E}_2, \vec{E}_3$ belong to a “distinguished” three-dimensional vector space S . For a fixed values of the coordinates τ, Z^1, Z^2, Z^3 the corresponding point $a(\tau, Z^\alpha)$ determines the equivalence class $A_{[a(\tau, Z^\alpha)]}$ of the equivalence relation

(C.2). The points from this equivalence class form a three-dimensional hyperplane in A which is parallel to S and after restricting the coordinate τ in the coordinate system (C.6) the remaining coordinates (Z^1, Z^2, Z^3) define the three-dimensional coordinate system parameterizing $A_{[a(\tau, Z^\alpha)]}$:

$$R^3 \ni (Z^1, Z^2, Z^3) \rightarrow [b + \tau \vec{L}] + Z^\alpha \vec{E}_\alpha = a(\tau, Z^\alpha) \in A_{[a(\tau, Z^\alpha)]},$$

$$\alpha = 1, 2, 3. \quad (\text{C.7})$$

Next, it is possible to repeat the reasoning from Appendix B and introduce the “three-dimensional” complete derivative “ ∇ ”. In general, it can be defined for arbitrary tensor fields and after taking the three-dimensional complete derivative “ D ” arbitrary tensor field gets an additional index from S^* . If the tensor field under consideration has at least one index from S then it is possible to define the corresponding “spatial divergence” and such “spatial divergence” can be denoted by a symbol “ div ”. It should be stressed that in the “amorphous Galilean space-time” there does not exist any “scalar product” and therefore the above introduced spatial divergence “ div ” has nothing common with “Euclidean divergence”.

In Appendix A the automorphisms of the affine space have been discussed and automorphisms with the affine straight lines as the sets of fixed points have been discussed in more detail. Now, in the amorphous Galilean space-time, in the parametric description of straight line

$$R \ni \tau \rightarrow a + \tau \vec{L} \in A, \quad \vec{L} \in T_A, \vec{L} \neq \vec{0}, \quad (\text{C.8})$$

one can distinguish “spatial” straight lines (with $\vec{L} \in S$) and “non-spatial” (with $\vec{L} \notin S$). For our purposes, only the case $\vec{L} \notin S$. As automorphisms of the amorphous Galilean space-time one can define such automorphisms of the four-dimensional affine space that “preserve the structure” of the amorphous Galilean space-time. Therefore, the results of Appendix A remain valid under the condition that a “directional vector” \vec{L} of a “non-spatial” straight line is supplemented to the basis in T_A by a basis $\{\vec{E}_1, \vec{E}_2, \vec{E}_3\}$ in S . In particular, the counterpart of (A.39) is now

$$f(b + \tau \vec{L} + Z^\alpha \vec{E}_\alpha) = b + \tau \vec{L} + Z^\alpha A_\alpha^\gamma \vec{E}_\gamma \quad (\text{C.9})$$

and it shows that for fixed b, \vec{L} the admissible automorphisms are determined up to a three-dimensional, nonsingular matrix A_α^γ , $\alpha, \beta = 1, 2, 3$.

In notation applied in (A.42), the counterpart of (C.9) is

$$f_{b, \vec{L}} \left(b + \tau \vec{L} + Z^\alpha \vec{E}_\alpha \right) = b + \tau \vec{L} + Z^\alpha A_\alpha^\gamma \vec{E}_\gamma. \quad (\text{C.10})$$

We shall see later the analogous expressions for “Galilean space-time with measurable time distances” and for a “real” Galilean space-time.

Appendix D

The Galilean space-time with measurable time intervals

This structure has been introduced in [1] on p. 383. It is more realistic model of “physical” Galilean space-time than the amorphous Galilean space-time (discussed in Appendix C) and without this model it is not possible to understand the invariant nature of the non-relativistic substantial derivative. Now the four-dimensional affine space A has the additional structure defined by a non-zero form Ψ from T_A^* , where T_A^* is the space of linear functionals on T_A . Therefore, it can be written as

$$(A, T_A, -, \Psi). \quad (\text{D.1})$$

For (D.1), the “measurable time interval” between the points $a, a' \in A$ is defined as

$$\text{dist}(a, a') = \left| \left\langle \Psi, \overrightarrow{a - a'} \right\rangle \right|, \quad (\text{D.2})$$

(see p. 383 of [1]). In the case discussed in Appendix C the vector subspace S belonging to the translation space T_A has been explicitly defined but now it is defined as the kernel of Ψ :

$$S = \{ \vec{z} \in T_A; \langle \Psi, \vec{z} \rangle = 0 \}. \quad (\text{D.3})$$

The important definition is the following:

$$W = \{ \vec{z} \in T_A; \langle \Psi, \vec{z} \rangle = 1 \}. \quad (\text{D.4})$$

This definition is a generalization of the definition (1.3) but now the metric aspect is absent. One can distinguish vector fields on A taking the values in the set W and we shall denote them as $\overrightarrow{c(a)}$:

$$A \ni a \rightarrow \overrightarrow{c(a)} \in W. \quad (\text{D.5})$$

We shall see that the vector fields $\overrightarrow{c(a)}$ of the form (D.5) have properties similar to the “non-relativistic four-velocities” $\overrightarrow{c(g)}$ from the standard Galilean space-time and therefore they are also called “non-relativistic four-velocities”. The different character of the domains of these fields is taken into account by using a different letter for their arguments.

In Appendix C the quantities $D\overrightarrow{B(a)}$ and $\nabla\overrightarrow{B(a)}$ have been defined for the vector field $\overrightarrow{B(a)}$. Now, those definitions can be applied also to the vector field $\overrightarrow{c(a)}$ and this vector field has an important particular property that $D\overrightarrow{c(a)}$ belongs to $S \otimes T_A^*$;

$$D\overrightarrow{c(a)} \in S \otimes T_A^*. \quad (\text{D.6})$$

The statement (D.6) can be proved because Observations 1 and 2 from the first chapter can be easily generalized to this case. Therefore also in the present case $(W, S, -)$ is the affine space. In analogy to the “complete” Galilean space-time (discussed in Introduction) one can define an affine atlas on $(W, S, -)$. It consists the coordinate systems of the form

$$R^3 \ni (Z^1, Z^2, Z^3) \rightarrow \overrightarrow{w} + Z^\alpha \vec{E}_\alpha \in W, \quad (\text{D.7})$$

where $\overrightarrow{w} \in W$ and $\vec{E}_\alpha, \alpha = 1, 2, 3$ is a basis in the vector space S . It is worth to mention that for the “complete” Galilean space-time, discussed in Introduction, the set of non-relativistic four-velocities was not only the affine space but it has the additional structure of an Euclidean space (because of the scalar product in S). Therefore, it was possible to introduce in (1.22) an orthonormal basis $\vec{e}_\alpha, \alpha = 1, 2, 3$. Now one does not have such a possibility but still there is an analogue for the formula (1.23) and it has a form

$$A \ni a \rightarrow \overrightarrow{c(a)} = \overrightarrow{w} + Z^\alpha(a) \vec{E}_\alpha \in W. \quad (\text{D.8})$$

After taking a complete derivative D of the function $\overrightarrow{c(a)}$ and taking into account the elementary properties of the complete derivative (see [1, 11]) one can see that

$$D\overrightarrow{c(a)} = D \left[\overrightarrow{w} + Z^\alpha(a) \vec{E}_\alpha \right] = D \left[Z^\alpha(a) \vec{E}_\alpha \right] = \vec{E}_\alpha \otimes DZ^\alpha(a). \quad (\text{D.9})$$

But in the present case $\vec{E}_\alpha, \alpha = 1, 2, 3$ is a basis in S and $DZ^\alpha(a)$ take the values in T_A^* and therefore the property (D.6) has been shown.

Now the general definition of a “four-dimensional divergence of a vector field $\overrightarrow{B(a)}$ ” (denoted in Appendix B as $Div \overrightarrow{B(a)}$), can be applied to the vector field $\overrightarrow{c(a)}$ and the result shall be denoted $Div \left\{ \overrightarrow{c(a)} \right\}$.

One can also check that $\nabla \overrightarrow{c(a)}$ is a tensor field taking the values in $S \otimes S^*$,

$$\nabla \overrightarrow{c(a)} \in S \otimes S^* \quad (\text{D.10})$$

and therefore it is possible to define the contraction of $\nabla \overrightarrow{c(a)}$, the corresponding contraction shall be denoted as $\div \left\{ \overrightarrow{c(a)} \right\}$:

$$\text{div} \left\{ \overrightarrow{c(a)} \right\} = \text{Tr}_{1,2} \nabla \overrightarrow{c(a)}. \quad (\text{D.11})$$

It is easy to see that

$$\text{Div} \left\{ \overrightarrow{c(a)} \right\} = \text{div} \left\{ \overrightarrow{c(a)} \right\}. \quad (\text{D.12})$$

As it has been already mentioned in Appendix B, the notion of the directional derivative on the affine space is defined in Schwartz's monograph [11] as the contraction of the complete derivative and the vector field. Also for the present case of the "Galilean space-time with measurable time intervals" one can distinguish the directional derivatives along the four-velocity field $\overrightarrow{c(a)}$ and the following notation

$$\frac{D}{Dt} = \overrightarrow{c(a)} \odot D \quad (\text{D.13})$$

can be introduced (here " \odot " denotes the contraction taken with respect to the corresponding arguments).

In Appendices A and C, the automorphisms of the affine space and of the "amorphous" Galilean space-time has been discussed shortly and the particular attention has been paid to automorphisms with the straight lines as the sets of fixed points. In particular, in Appendix C sets of fixed points correspond to the "non-spatial" straight lines. Now it is possible to introduce an additional restriction; instead of arbitrary "non-spatial" directional vectors of the straight line one can choose the straight lines of the form

$$R \ni t \rightarrow b + t\overrightarrow{w} \in A, \quad \overrightarrow{w} \in W, \quad (\text{D.14})$$

with W given by (D.4). A counterpart of expressions (A.39) and (C.9) is now:

$$f \left(b + \tau\overrightarrow{w} + Z^\alpha \overrightarrow{E}_\alpha \right) = b + \tau\overrightarrow{w} + Z^\alpha A_\alpha^\gamma \overrightarrow{E}_\gamma. \quad (\text{D.15})$$

In turn, a counterpart of (A.42) and (C.10) is now

$$f_{b, \overrightarrow{w}} \left(b + \tau\overrightarrow{w} + Z^\alpha \overrightarrow{E}_\alpha \right) = b + \tau\overrightarrow{w} + Z^\alpha A_\alpha^\gamma \overrightarrow{E}_\gamma. \quad (\text{D.16})$$

It is worth to mention that the “directional vector” in the parametric description of the straight line (D.14) cannot be multiplied by a negative number (in other words, the straight lines are oriented). For fixed b and \vec{w} and for a fixed basis $\vec{E}_1, \vec{E}_2, \vec{E}_3$ in S our automorphisms are determined up to a three-dimensional, nonsingular matrix A_α^γ , $\alpha, \beta = 1, 2, 3$. In Appendix F, we shall see analogous expressions for a “real” Galilean space-time.

Appendix E

Direct sum of affine spaces and the Boltzmann kinetic equation

In Chapter five, Boltzmann equation was introduced in the approximate way, sufficient for derivations of “invariant” moment equations. Its solutions are the real functions on the product $G \times W$ and at the beginning of the Chapter five it is shown that that product has a canonical structure of the seven-dimensional affine space. On G there exists a canonical complete derivative (denoted as “ D ”) and on W there exists a canonical complete derivative (denoted as “ ∇ ”). The equation (5.2) for functions on $G \times W$ is defined in the terms of the differential operators “ D ” and “ ∇ ”. However, (5.2) is not a complete solution of our problem of writing a corresponding one-particle Liouville equation in an invariant form. On $G \times W$ there exists a canonical “seven-dimensional” complete derivative but it is not identical with the pair of operators D, ∇ .

In order to discuss that in more detail let us consider two finite-dimensional affine spaces, $(A, T_A, -)$ and $(B, T_B, -)$, $\dim A = \dim T_A = 4$, $\dim B = \dim T_B = 3$. As it has been already written at the beginning of Chapter five, a product $A \times B$ is an seven-dimensional affine space and the operation “ $-$ ” for this product is defined “naturally”; for $a, a' \in A$ and $b, b' \in B$, one can consider two pairs (a, b) , (a', b') from $A \times B$ and then the “rule for subtraction” is

$$(a, b) - (a', b') = (a - a', b - b') \in T_A \oplus T_B. \quad (\text{E.1})$$

In (E.1), $T_A \oplus T_B$ denotes a direct sum of the spaces T_A and T_B (the definition of a direct sum is given, for example, in [20] on p. 27). Now we want to discuss the notion of a complete derivative on the product of affine spaces. Of course, since a product of affine spaces is also an affine space and every finite-dimensional affine space is endowed with a canonical complete derivative the existence of this

derivative is trivial. However, in Chapter five the Boltzmann equation has been written in terms of the pair of complete derivatives (D, ∇) , and now we want to know when the complete derivative on $G \times W$ can be written in that way. Let us start with a definition of the affine coordinate system on $A \times B$. According to a general rule, one has to choose an arbitrary basis from $T_A \oplus T_B$ and repeat procedure described in Appendices A and B. However, such an approach does not lead to anything similar to the l.h.s. of Boltzmann equation. The reason for that is that not all bases are allowed if one wants to have a link to the standard form of Boltzmann equation.

Among all bases in $T_A \oplus T_B$ it is possible to choose such bases in $T_A \oplus T_B$ that are composed of a pairs, containing a basis in T_A and a basis in T_B . In general, a basis in T_A contains four linearly independent vectors from T_A (denoted as $\{\vec{E}_1, \vec{E}_2, \vec{E}_3, \vec{E}_4\}$) and a basis in T_B contains three linearly independent vectors from T_B (denoted as $\{\vec{U}_1, \vec{U}_2, \vec{U}_3\}$). Then $\{\vec{E}_1, \vec{E}_2, \vec{E}_3, \vec{E}_4, \vec{U}_1, \vec{U}_2, \vec{U}_3\}$ is a basis in $T_A \oplus T_B$. The corresponding affine coordinate systems in $A \times B$ are defined explicitly according to the rule

$$(\lambda^1, \lambda^2, \lambda^3, \lambda^4, \eta^1, \eta^2, \eta^3) \rightarrow (a, b) + \sum_{i=1}^{i=4} \lambda^i \vec{E}_i + \sum_{i=1}^{i=3} \eta^i \vec{U}_i =$$

$$\left(a + \sum_{i=1}^{i=4} \lambda^i \vec{E}_i, b + \sum_{i=1}^{i=3} \eta^i \vec{U}_i \right) = (a', b') \in A \times B. \quad (\text{E.2})$$

As it has been already mentioned, on a seven-dimensional affine space $A \times B$ there exists a canonical “seven-dimensional” complete derivative and it can be denoted “ D' ”. Let us denote a canonical complete derivative on A as “ D_A ” and let us denote a canonical complete derivative on B as “ ∇_B ”. By a direct inspection it is possible to check that only for particular bases in $T_A \oplus T_B$, composed of a basis in T_A and a basis in T_B , the following identification takes place

$$D' = (D_A, \nabla_B). \quad (\text{E.3})$$

In the construction of the l.h.s. of the Boltzmann equation one takes the basis in $T_G \oplus S$ of the above mentioned kind. Moreover, the basis in T_G is restricted to the basis of the form $\{\vec{w}, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and the basis in S is restricted to the basis of the form $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ (with $\vec{e}_1, \vec{e}_2, \vec{e}_3$ forming an orthonormal basis in S). Here, we have no place for a detailed discussion but such a discussion should involve also a “symplectic” aspect of the Liouville operator. The r.h.s. of Boltzmann equation also deserves an analysis but that is outside the scope of the present text.

Appendix F

On the Galilean invariance of fluid mechanics and the non-relativistic kinetic theory

As it has been already mentioned, the “principle of material indifference” is not consistent with the kinetic theory of gases; for example, the kinetic theory of gases leads to the non-objective macroscopical constitutive laws for such quantities as the heat flux [13]. Other critical remarks concerning the principle of material indifference are given in [55] and in Rymarz’s textbook [44] on p. 158.

It is possible that this discrepancy can be eliminated after formulating the fluid mechanics and the kinetic theory in the manner invariant with respect to the automorphisms of the Galilean space-time. In particular, it would be interesting to find such subgroups of automorphisms of the Galilean space-time that impose similar restrictions on constitutive functions to those corresponding to the “principle of material objectivity” [12, 44].

In order to look for for such subgroups we make use of the scheme proposed by Sławianowski [1] and investigate different “models” of Galilean space-time.

The simplest “model” is a four-dimensional affine space and it is discussed in Appendix A. Other models are the “amorphous Galilean space-time” (discussed in Appendix C) and the “Galilean space-time with measurable time intervals” (discussed in Appendix D). The “real” Galilean space-time is discussed in the main text. All these structures are algebraic structures composed of affine spaces with additional structures. For all these structures it is possible to define groups of their affine automorphisms (see (A.11)). All these structures form transformation groups. A general theory of transformation groups has been formulated by Rychlewski [45]. In his general formulation, an arbitrary group acts in a group

of bijections of an arbitrary set but on p. 70 of [45] a possibility of imposing additional structures in the theory is described. For the cases analyzed in the present text, discussed Γ -structures are groups of affine bijections acting in the finite-dimensional affine spaces (and preserving the additional structures present in those affine spaces).

In this text we have no space for a detailed discussion of these aspects and for our main aim here is to give a short formulation the hypothesis that affine automorphisms with the straight lines as the sets of fixed points are mainly responsible for the required symmetry properties of constitutive functions.

Therefore, in Appendix A the explicit form of affine automorphisms “preserving” an affine straight line has been determined (see (A.39) and (A.42)). That solution can be easily adopted to the cases of “amorphous Galilean space-time” and “Galilean space-time with measurable time distances” (see (C.9), (C.10) and (D.15), (D.16)). The case of Galilean space-time with measurable time distances is particularly instructive. In its translation space there is a distinguished set of vectors, denoted as W (see (D.4)). Vectors from this set are called the “non-relativistic four velocities” because their properties are similar to the non-relativistic four-velocities from the standard Galilean space-time. The explicit form of the corresponding automorphisms (D.16) is:

$$f_{b, \vec{w}} \left(b + \tau \vec{w} + Z^\alpha \vec{E}_\alpha \right) = b + \tau \vec{w} + Z^\alpha A_\alpha^\gamma \vec{E}_\gamma, \quad (\text{F.1})$$

where $A_\alpha^\gamma, \alpha, \gamma = 1, 2, 3$ are non-singular matrices. The important property of the mapping (F.1) is that for

$$Z^\alpha = 0$$

this mapping reduces to

$$f_{b, \vec{w}} (b + \tau \vec{w}) = b + \tau \vec{w}.$$

The another important fact is that (F.1) can be reduced to the “spaces of simultaneous events”, that is, it leaves the value of the variable τ unchanged.

By means of the example (F.1), one can easily write a similar expression for a “complete” Galilean space-time.

All structures from the Galilean space-time with measurable time distances are present in the “complete” Galilean space-time. However, in Galilean space-time one has a scalar product $(,)$ in S (compare Introduction, p. 2) and therefore it is possible to choose an orthonormal basis $\vec{e}_\alpha, \alpha = 1, 2, 3$ in S . The vectors of such basis satisfy a condition

$$(\vec{e}_\alpha, \vec{e}_\beta) = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3, \quad (\text{F.2})$$

with $\delta_{\alpha\beta}$ being Kronecker delta.

One can parameterize the Galilean space-time using affine coordinate systems with bases in T_G determined by a single vector from W and orthonormal reper $\vec{e}_1, \vec{e}_2, \vec{e}_3$ in S and in this way we arrive again at the notion of the “inertial coordinate system” (described in Introduction) but at this time one can see its relation to affine automorphisms. Therefore it is possible to impose on our automorphisms an additional condition that they have to preserve a distance between the points from the spaces of simultaneous events. The result can be obtained after modifying (F.1); after writing “ t ” instead “ τ ” and “ x^α ” instead “ Z^α ”, and after inserting Θ_α^γ in place of A_α^γ one arrives at the following identity:

$$f_{b, \vec{w}}(b + t\vec{w} + x^\alpha \vec{e}_\alpha) = b + t\vec{w} + x^\alpha \Theta_\alpha^\gamma \vec{e}_\gamma. \quad (\text{F.3})$$

It can be easily checked that Θ_α^γ are orthogonal matrices and therefore after denoting

$$x^\alpha \vec{e}_\alpha = \vec{r}, \quad (\text{F.4})$$

(F.3) can be written in the form

$$f_{b, \vec{w}}(g + t\vec{w} + \vec{r}) = g + t\vec{w} + \Theta \vec{r}, \quad (\text{F.5})$$

where Θ is an orthogonal tensor and g is an arbitrary point of G . Let us denote the set of “affine” automorphisms of Galilean space-time for which the set of fixed points is the affine straight line

$$R \ni t = g + t\vec{w} \quad (\text{F.6})$$

as $\mathcal{A}(g, \vec{w}, \Theta)$; then the mapping (F.5) can be written alternatively in the following way:

$$\mathcal{A}(g, \vec{w}, \Theta) * (g + t\vec{w} + \vec{r}) = g + t\vec{w} + \Theta \vec{r}. \quad (\text{F.7})$$

For any space of simultaneous events $H_{[g']}$, $g' \in G$, (F.5) transforms points belonging to $H_{[g']}$ into other points belonging to $H_{[g']}$.

As it has been already discussed, the “affine” formalism for Galilean space-time formulated in terms of the inertial atlas is equivalent to the “standard” formalism, formulated in terms of the observations of an inertial observer. The “transformation rule” between both descriptions takes into account the explicit parametric expression for the world line of an inertial observer and an inertial observer with the world line (F.6) parameterizes the space-time around him according to the rule

$$(R, S) \ni (t, \vec{r}) \rightarrow g + t\vec{w} + \vec{r}. \quad (\text{F.8})$$

For a given inertial observer, identified with his affine (non-spatial) world line (F.6) the action of the orthogonal mapping Θ can be written as

$$\Theta * (t, \vec{r}) = (t, \Theta \vec{r}). \quad (\text{F.9})$$

For a given value of the time coordinate (F.9) becomes the standard action of the orthogonal group on the Euclidean vector space. This group, together with its action on Euclidean tensors, is applied in theory of constitutive functions ([12, 13, 15, 23, 24, 25, 40, 41, 44, 45]) what motivates our hypothesis that the explicit condition for restricting field equations can be requirement of “form invariance” with respect to the “affine” automorphisms of Galilean space-time. Equivalent formulations in terms of the point symmetries are also possible. For a detailed proof, it is necessary to define the action of “affine” automorphisms on the tensor fields on affine spaces and this is outside the scope of the present text. In this context, one can see [11] and [45]; it is worth to remark that interesting cases of Γ -structures in a sense of Rychlewski exist on the normed affine spaces and all “models” discussed in the present text are particular cases of those Γ -structures.

Obviously, similar automorphisms can be investigated for non-relativistic phase densities.

As it has been already mentioned, the non-relativistic distribution function is defined on $G \times W$ and the corresponding set of pairs $(g, \vec{w}) \in G \times W$ can be parameterized in the following inertial coordinates

$$(t, x^i, u^j) \rightarrow (g_0 + t\vec{w}_I + x^i \vec{e}_i, \vec{w}_I + u^j \vec{e}_j) \in G \times W. \quad (\text{F.10})$$

It should be remembered that such a parametrization is defined for a fixed world line, which is defined by a parametric equation of a straight line

$$R \ni t \rightarrow g_0 + t\vec{w}_I, \quad \vec{w}_I \in W, \quad (\text{F.11})$$

and that parametric description of a world line (F.11) is not unique; if one considers a “non-spatial”, straight line in G “as a set of points” then its direction is defined uniquely by a vector from W but the parametrization in terms of the “absolute time” is not unique. A given non-spatial world line can be bijectively parameterized in terms of the set $[G]$, ($[G]$ is a set of “absolute time” instants, which carry a structure of an oriented one-dimensional affine space) but in order to parameterize it in terms of the real numbers one has to choose a time instant with “time equal to zero” and that time instant is determined by a choice of a point g_0 .

Usually, one prefers vector notation for euclidian” objects and therefore is useful to denote

$$x^\alpha \vec{e}_\alpha = \vec{r}, \quad u^j \vec{e}_j = \vec{u}, \quad (\text{F.12})$$

then the domain of a non-relativistic distribution function can be parameterized in the form

$$(t, \vec{r}, \vec{u}) \rightarrow (g_0 + t\vec{w}_I + \vec{r}, \vec{w}_I + \vec{u}) \in G \times W. \quad (\text{F.13})$$

It should be stressed that the transition to the vector calculus for euclidian objects is very convenient here because in this way one can eliminate matrix notation from our automorphisms. Now it is possible to define the action of $\mathcal{A}(g, \vec{w}, \Theta)$ also on the domain of the non-relativistic distribution function; for $g_0 \in G, \vec{w}_I \in W$ and arbitrary orthogonal mapping Θ the action of $\mathcal{A}(g_0, \vec{w}_I, \Theta)$ on $G \times W$ is defined as:

$$\begin{aligned} \mathcal{A}(g_0, \vec{w}_I, \Theta) * (g, \vec{w}) &= \mathcal{A}(g_0, \vec{w}_I, \Theta) * (g_0 + t\vec{w}_I + \vec{r}, \vec{w}_I + \vec{u}) = \\ & (g_0 + t\vec{w}_I + \Theta\vec{r}, \vec{w}_I + \Theta\vec{u}) \end{aligned} \quad (\text{F.14})$$

In order to how does it work on a distribution function, one can use the notation introduced in (1.32) and it is possible to check that

$$\begin{aligned} \mathcal{A}(g_0, \vec{w}_I, \Theta) * f(g, \vec{w}) &= \mathcal{A}(g_0, \vec{w}_I, \Theta) * f(g_0 + t\vec{w}_I + \vec{r}, \vec{w}_I + \vec{u}) = \\ \mathcal{A}(g_0, \vec{w}_I, \Theta) * f_{g_0, \vec{w}_I}(\vec{r}, \vec{u}) &= f_{g_0, \vec{w}_I}(\Theta\vec{r}, \Theta\vec{u}). \end{aligned} \quad (\text{F.15})$$

A rigorous proof can be made along the lines given in [45] but our aim here is only to formulate an intuitive hypothesis on the alternative origin of “non-relativistic constitutive functions”. We believe that the Boltzmann equation is “form invariant” with respect to “affine” automorphisms of the Galilean space-time and that important class of such automorphisms are the mappings of the form $\mathcal{A}(g_0, \vec{w}_I, \Theta)$. More results on Galilean invariance shall be given in [87].

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Bibliography

- [1] J. J. Sławianowski, *Geometry of Phase Spaces*, John Wiley & Sons, Chichester, PWN, Warszawa, 1991.
- [2] S. Piekarski, Quasi-particle kinetic equation in a deformable material continuum, *Arch. Mech.*, 43, 6, 769–782, 1991.
- [3] S. Piekarski, On the invariant formulation of fluid mechanics, <http://xxx.lanl.gov/ftp/physics/papers/0411/0411154.pdf>
- [4] S. Piekarski, On integration of constraints imposed on a system of conservation laws by the second law of thermodynamics, *Continuum Thermodyn.*, 4, 109–119, 1992.
- [5] Z. Peradzyński, *Geometry of Interactions of Riemann Waves: Advances in Nonlinear Waves*, Ed.: L. Debneth, Pitman, 244–283, 1985.
- [6] Z. Peradzyński, *Geometry of Nonlinear Interactions in Partial Differential Equations*, Habilitation Thesis, Warsaw, 1981 (in Polish).
- [7] S. Piekarski, Geometrical aspect of symmetric conservative systems of partial differential equations, *Arch. Mech.*, 44, 603–614, 1992.
- [8] S. Piekarski, Geometrical aspect of symmetrization of quasilinear systems of the first-order partial differential equations, *Arch. Mech.*, 45, 3, 369–377, 1993.
- [9] S. Piekarski, On the uniqueness of symmetric algebraic consequences implied by a system of conservation laws consistent with the additional conservation equation, *Cont. Mech. Thermod.*, 6, 21–29, 1994.
- [10] S. Piekarski, Invariant Description of Symmetric Conservative Systems, *Arch. Mech.*, 46, 1-2, 105–120, 1994.
- [11] L. Schwartz, *Analyse Mathématique*, Hermann, 1967.

-
- [12] S. Jemioło, J. J. Telega, Representations of tensor functions and applications in continuum mechanics, IFTR Reports 3/4, 1997.
 - [13] K. Wilmański, Thermomechanics of continua, Berlin, Springer, 1998.
 - [14] J. J. Sławianowski, Geometry of Phase Spaces, PWN, Warszawa, 1975, (in Polish).
 - [15] J. Sławianowski, Analytical Mechanics of Deformable Bodies, PWN, Warszawa, 1982 (in Polish).
 - [16] S. Piekarski, On the Navier–Stokes equations for water, Archives of Acoustics, 31, 2, 265–271, 2006.
 - [17] S. Piekarski, On the classification of dense fluids, Archives of Acoustics, 32, 4, 933–940, 2007.
 - [18] G. B. Whitham, Linear and nonlinear waves, Wiley-Interscience Publications, John Wiley & Sons, New York, 1974.
 - [19] K. Huang, Statistical Mechanics, John Wiley & Sons, New York, 1963.
 - [20] J. Komorowski, From the complex numbers to the tensors, spinors, Lie algebras and quadrics, PWN, Warszawa, 1978 (in Polish).
 - [21] B. F. Schultz, Geometrical methods of mathematical physics, Cambridge University Press, Cambridge, 1982.
 - [22] P. Wintgen, R. Sulanke, Differential geometry and the theory of Fibre Bundles, PWN, Warszawa 1978 (in Polish); translated from: Differentialgeometrie und Faserbündel, Berlin, 1972.
 - [23] C. Truesdell, A first course in rational continuum mechanics, Vol. 1, 2nd ed., Academic Press, San Diego, 1991.
 - [24] W. Kosiński, Field singularities and wave analysis in continuum mechanics, Warszawa, PWN; Chichester, E. Horwood, 1986.
 - [25] C. C. Wang, Mathematical methods of mechanics and electromagnetism, Plenum Press, New York, 1979.
 - [26] W. Kondracki, J. Rogulski, Geometrical methods in classical mechanics and in classical field theory, Publications of Warsaw University, 1978 (in Polish).

-
- [27] M. Heller, *Theoretical foundations of cosmology*, PWN, Warszawa, 1975 (in Polish).
- [28] W.D. Curtis, F.R. Miller, *Differential manifolds and theoretical physics*, Academic Press, London, 1985.
- [29] B.A. Dubrovin, S.P. Hovikov, A.T. Fomenko, *Contemporary Geometry. Methods and Applications*, Nauka, Moskva, 1986 (in Russian).
- [30] B. Baranowski, A.E. de Vries, A. Haring, *Thermal diffusion in systems with some transformable components*, *Advances in Chemical Physics*, Eds.: I. Prigogine, S.A. Rice, John Wiley & Sons, London, 1969.
- [31] B.W. Alekseyev, *Matematitsheskaya kinetika reagiriyushtshich gazov*, Nayka, Moskva, 1982 (in Russian).
- [32] J.F. Clarke, *Physico-chemical gas dynamics*, *Kinetic Theory and Gas Dynamics*, Ed.: Cercignani, CISM Courses and Lectures, no. 293, Springer, 1988.
- [33] C. Cercignani, R. Illner, M. Pulverenti, *The Mathematical Theory of Dilute Gases*, Springer Verlag, New York, 1994.
- [34] A. Trautman, W. Kopczyński, *Spacetime and gravitation*, PWN, Warszawa 1981 (in Polish).
- [35] S. Piekarski, *On the modified Fick law and its potential applications*, *Journ. Tech. Phys.*, 44, 2, 125–131, 2003.
- [36] S. Chapman, T.G. Cowling, *The mathematical theory of non-uniform gases*, Cambridge University Press, 1960.
- [37] S. Piekarski, *On diffusion and thermodiffusion in a gravity field*, *Journ. Tech. Phys.*, 44, 3, 329–337, 2003.
- [38] S. Piekarski, *Stress-assisted diffusion and the modified Fick law*, *Journ. Tech. Phys.*, 46, 1, 3–7, 2005.
- [39] A. Kawczyński, *Chemical reactions from equilibrium to dissipative structures and chaos*, Wyd. Naukowo-Techniczne, Warszawa, 1990 (in Polish).
- [40] Z. Wesółowski, Cz. Woźniak, *Fundamentals of nonlinear theory of elasticity*, PWN, Warszawa, 1970 (in Polish).

-
- [41] K. Wilmański, *Thermodynamical Models of Continuous Media*, Poznań Technical University, 1985 (in Polish).
- [42] J. H. Dymond, E. B. Smith, *The Virial Coefficients of Pure Gases and Mixtures, A Critical Compilation*, Clarendon Press, Oxford, 1980.
- [43] *Contemporary Trends in Thermodynamics*, Polish Academy of Sciences, Wydział IV Nauk Technicznych, Komitet Termodynamiki i Spalania, Eds.: Z. Bilicki, J. Mikielwicz, S. Sieniutycz, Wydawnictwo Instytutu Maszyn Przepływowych, Gdańsk, 2001 (in Polish).
- [44] Cz. Rymarz, *Mechanics of Continuous Media*, Wydawnictwo Naukowe PWN, Warszawa, 1993 (in Polish).
- [45] J. Rychlewski, *Symmetry of Causes and Results*, Wydawnictwo Naukowe PWN, Warszawa, 1991 (in Polish).
- [46] J. Rychlewski, *Dimensions and Similarity*, Wydawnictwo Naukowe PWN, Warszawa 1991 (in Polish).
- [47] *Nonlinear Acoustics*, Eds.: M. F. Hamilton, D. T. Blackstock, Chapter: "Biomedical Applications", Academic Press, 1998.
- [48] J. H. Dymond, E. B. Smith, *The Virial Coefficients of Pure Gases and Mixtures. A Critical Compilation*. Clarendon Press, Oxford, 1980.
- [49] G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, 1967, 2000.
- [50] S. Piekarski, On the Modified Fick Law and its potential Applications, *Journal of Technical Physics*, 44, 2, 125–131, 2003.
- [51] S. Piekarski, On Diffusion and Thermodiffusion in a Gravity Field, *Journal of Technical Physics*, 44, 3, 329–337, 2003.
- [52] S. Piekarski, On the Debye effect in a semiconductor and an electrolyte, arXiv:cond-mat/0403758v2 2Apr 2004.
- [53] S. Piekarski, Stress-Assisted Diffusion and the Modified Fick Law, *Journal of Technical Physics*, 46, 1, 3–7, 2005.
- [54] S. Piekarski, On the Thermodiffusion Equation for Electrically Charged Matter, *Journal of Technical Physics*, 46, 2, 83–95, 2005.

-
- [55] L.H. Soderholm, On the Range of Validity of the Principle of Material Frame-Indifference, *Arch. Mech. Stos.*, 33, 1981.
- [56] Z. Banach, S. Piekarski, Irreducible Tensor Description. I. A Classical Gas, *J. Math. Phys.*, 30, 1804–1815, 1989.
- [57] J. Wójcik, Conservation of energy and absorption in acoustic fields for linear and nonlinear propagation, *J. Acoust. Soc. Am.*, 104, 5, November 1998.
- [58] Z. Banach, S. Piekarski, Perturbation theory based on the Einstein–Boltzmann system. I. Illustration of the theory for a Robertson–Walker geometry, *Journal of Mathematical Physics*, 35, 9, September 1994.
- [59] Z. Banach, S. Piekarski, Perturbation theory based on the Einstein–Boltzmann system. II. Illustration of the theory for an almost-Robertson–Walker geometry, *International Journal of Mathematical Physics*, 35, 11, November 1994.
- [60] Z. Banach, S. Piekarski, An almost–Robertson–Walker universe model and the equivalence classes of perturbations: Nonbarotropic perfect fluids, *Ann. Inst. Henri Poincaré*, Vol. 65, no. 3, 1996.
- [61] Z. Banach, S. Piekarski, Gauge-invariant cosmological perturbation theory for collisionless matter: Application to the Einstein–Liouville system, *Gen. Relativ. Gravit.*, 28, 1996.
- [62] Z. Banach, S. Piekarski, Theory of Cosmological Perturbations Formulated in Terms of a Complete Set of Basic Gauge-Invariant Quantities, *International Journal of Theoretical Physics*, Vol. 35, No. 3, 1996.
- [63] Z. Banach, S. Piekarski, Geometrization of Linear Perturbation Theory for Diffeomorphism-Invariant Covariant Field Equations. I. The Notion of a Gauge-Invariant Variable. *International Journal of Theoretical Physics*, Vol. 36, No. 8, 1997.
- [64] Z. Banach, S. Piekarski, Geometrization of Linear Perturbation Theory for Diffeomorphism-Invariant Covariant Field Equations. II. Basic Gauge-Invariant Variables with Applications to de Sitter Space-Time, *International Journal of Theoretical Physics*, Vol. 36, No. 8, 1997.
- [65] Z. Banach, S. Piekarski, Equivalence classes of perturbations in cosmologies of Bianchi types I and V: Propagation and constraint equations. *Journal of Mathematical Physics*, Vol. 41, No. 10, October 2000.

- [66] A. M. Vinogradov, L. S. Krasilshchik and V. V. Lichagin, Introduction to geometry of nonlinear differential equations, Moscow Science, 1986 (in Russian).
- [67] T. Kujawska, Investigations of nonlinear properties of biological media by means of ultrasonic waves, IFTR REPORTS, 9/2006 (in Polish).
- [68] A. Wawrzyńczyk, Group Representations and Special Functions, Examples and problems prepared by A. Strasburger, D. Reidel, Boston, PWN, Warszawa, 1984.
- [69] A. W. Knap, Lie groups beyond an introduction, Birkhauser, Boston, 1996.
- [70] M. Hamermesh, Group Theory and Its Applications to Physical Problems, PWN, Warszawa, 1968 (in Polish).
- [71] K. Maurin, Chapter: Mathematics and Physics, Mathematical Lexicon, Eds.: M. Kordos, M. Skwarczyński, W. Zawadowski, Wiedza Powszechna, 1993 (in Polish).
- [72] S. K. Godunov, V. M. Gordienko, The Simplest Galilean-invariant and Thermodynamically Consistent Conservation Laws, Journal of Applied Mechanics and Technical Physics, Vol. 43, No. 1, pp. 1–12, 2002.
- [73] S. K. Godunov, V. M. Gordienko, Complicated Structures of Galilean-Invariant Conservation Laws, Journal of Applied Mechanics and Technical Physics, Vol. 43, No. 2, pp. 175–189, 2002.
- [74] R. M. Cherniha, Nonlinear Galilei-Invariant PDEs with Infinite-Dimensional Lie Symmetry, Journal of Mathematical Analysis and Applications, 253, 126–141, 2001.
- [75] A. M. Vinogradov, Local Symmetries and Conservation Laws, Acta Applicandae Mathematicae, 2, 21–78, 1984.
- [76] I. S. Krasilshchik, A. M. Vinogradov, Nonlocal Symmetries and the Theory of Coverings: An Addendum to A. M. Vinogradov’s “Local Symmetries and Conservation Laws”, Acta Applicandae Mathematicae, 2, 79–96, 1984.
- [77] W. D. McComb, Galilean invariance and vertex renormalization in turbulence theory, Physical Review E 71, 037301-4, 2005.
- [78] Ch. Tong, Galilean invariance of velocity probability density function transport equation, Physics of Fluids, Vol. 15, No. 7, 2073–2076, 2003.

-
- [79] J.-M. Levy-Leblond, Galilei Group and Nonrelativistic Quantum Mechanics, *Journal of Mathematical Physics*, Vol. 4, No. 6, 776–788, 1963.
- [80] Ch. G. Speziale, Comments on the “material frame-indifference” controversy, *Physical Review A*, Vol. 36, No. 9, 4522–4535, 1987.
- [81] D. Razafindralandy, A. Hamdouni, M. Oberlack, Analysis and development of subgrid turbulence models preserving the symmetry properties of the Navier–Stokes equations, *European Journal of Mechanics B/Fluids*, 26, 531–550, 2007.
- [82] D. Razafindralandy, A. Hamdouni, C. Be’ghein, A class of subgrid-scale models preserving the symmetry group of Navier–Stokes equations, *Communications in Nonlinear Science and Numerical Simulation*, 12, 243–253, 2007.
- [83] D.-W. Chiou, Galileo symmetries in polymer particle representation, *Class. Quantum Grav.*, 24, 2603–2620, 2007.
- [84] G. Manno, F. Oliveri, R. Vitolo, On differential equations characterized by their Lie point symmetries, *J. Math. Anal. Appl.*, 332, 767–786, 2007.
- [85] G. A. Goldin, V. M. Shtelen, On Galilean invariance and nonlinearity in electrodynamics and quantum mechanics, *Physics Letters A* 279, 321–326, 2001.
- [86] N. Ch. Ibrahimov, *Gruppy preobrazovanij v matematič’eskoj fizike*, Nauka, Moskva, 1983.
- [87] S. Piekarski, On invariant harmonic analysis on the Galilean space-time, in preparation.

Streszczenie i elementy oryginalne rozprawy

Przedstawiona praca dotyczy sformułowania mechaniki cieczy w języku nierelatywistycznej czasoprzestrzeni Galileusza (jako struktury algebraicznej) oraz niezmienniczości operatorów różniczkowych i równań. Treść rozprawy jest kontynuacją wcześniejszych wyników, które otrzymano w roku 1991, gdy to Sławianowski wprowadził modele czasoprzestrzeni Galileusza („amorphous Galilean space-time” i „Galilean space-time with measurable time distances”) opisane w [1] oraz autor wprowadził w [2] pojęcie „nierelatywistycznej czteropędkości”.

Niektóre wyniki, opisane w rozprawie, zostały opublikowane przez autora niedawno [16, 17], natomiast na ostateczną treść miały wpływ zarówno wcześniejsze prace autora o niezmienniczych definicjach równań cząstkowych na przestrzeniach afinicznych i przestrzeniach afinicznych z dodatkowymi strukturami [3, 4, 7–10] jak i prace dotyczące niezmienników w ogólnorelatywistycznym rachunku perturbacyjnym, wykonane przez autora wspólnie z Z. Banachem [58–65].

Najwięcej uwagi poświęcono równaniom Naviera–Stokesa–Fouriera i ich konsekwencjom.

Z punktu widzenia zastosowań, najważniejsze wydają się wyniki dotyczące gęstych cieczy, które zostały opisane w „standardowej” notacji w rozdziale drugim (zostały one częściowo opublikowane w artykułach [16, 17]). „Tożsamości termostatyczne” otrzymuje się tam jako konsekwencje równań Naviera–Stokesa–Fouriera a jako pola pierwotne przyjmuje gęstość masy ρ i temperaturę T . Opisano tam propozycję autora, aby „gęstą ciecz” definiować poprzez warunek, że gęstość energii (na jednostkę masy) zależy nie tylko od temperatury T , ale także od gęstości masy ρ . Jeśli gęstość energii na jednostkę masy oznaczyć $E(\rho, T)$, to naturalne jest pytanie o przypadki pośrednie między „gęstymi cieczami” a gazami doskonałymi. W przedstawionej rozprawie (i w pracach [16, 17]) autor pokazał, że jeśli gęstość energii na jednostkę masy zależy tylko od temperatury oraz jednocześnie spełniona jest tożsamość Gibbsa, to ciśnienie jest dowolną funkcją od

gęstości masy mnożoną przez temperaturę T . Podano nowe rozwiązania dla tożsamości Gibbsa, opisujące w szczególności gęste cieczy i obliczono odpowiednie prędkości dźwięku. Pokazano, że w ramach zaproponowanego podejścia można opracować przybliżoną klasyfikację gęstych cieczy, która w szczególności może przypominać rozwinięcia wirialne i, dla przykładu, zbadano proste przypadki „gęstych cieczy”.

Modele gęstych cieczy mogą być użyteczne np. dla akustyki medycznej, gdzie często tkanki biologiczne są modelowane jako gęste cieczy. Przedstawione w rozprawie wyniki mają tylko zilustrować proponowane podejście i uzasadnić celowość kontynuowania badań.

Przy badaniu gęstych cieczy, Galileuszowskie niezmienniki pomogły uściślić dyskusję. Jednocześnie, Galileuszowska niezmienniczość operatorów różniczkowych i równań pola może być osobnym tematem badań i niezmiennicze zapisanie równań Naviera–Stokesa–Fouriera zostało ułatwione dzięki obserwacji autora, że „pochodna substancjalna” nierelatywistycznej hydrodynamiki kontinuum jest pochodną kierunkową w kierunku „nierelatywistycznej czteroprędkości” [3].

W przedstawionej rozprawie dyskutowane są też niektóre fakty, dotyczące niezmienniczych aspektów nierelatywistycznej teorii kinetycznej. We wstępie podane są niezmiennicze odpowiedniki funkcji rozkładu i „ μ -przestrzeni”. W rozdziale czwartym podano niezmiennicze sformułowanie nierelatywistycznej funkcji rozkładu (z włączeniem rozkładów kwantowych w przybliżeniu bezspinowym). W rozdziale piątym dyskutowane jest równanie Boltzmanna. Jednym z aspektów teorii kinetycznej są równania momentowe; ogólna postać równań momentowych dla równania Boltzmanna została opublikowana przez autora (wspólnie z Z. Banachem) w roku 1989 [56]. Jednak podane tam równania nie są zapisane poprzez niezmienniki i sposób, w jaki pojawiają się „niezmiennicze” momenty w nierelatywistycznej teorii kinetycznej jest krótko omawiany w rozdziale piątym.

Aby lepiej zrozumieć operatory różniczkowe stosowane przy niezmienniczym zapisie równań Naviera–Stokesa–Fouriera, w dodatkach opisujemy kanoniczne operatory różniczkowe na rozważanych „modelach” czasoprzestrzeni Galileusza. W dodatkach szczególną uwagę zwracamy też na te podgrupy automorfizmów rozważanych przestrzeni, dla których zbiorami punktów stałych są proste afiniczne. Mamy nadzieję, że badanie takich przekształceń dla czasoprzestrzeni Galileusza mogłoby pomóc przy dyskusji „zasady obiektywności materialnej”.

Warto podkreślić, że zarówno przy wprowadzaniu „nierelatywistycznych” niezmienników (opisywanych w przedstawionej pracy) jak i przy wprowadzaniu niezmienników dla ogólnorelatywistycznego rachunku perturbacyjnego (wprowadzonych wspólnie z Z. Banachem, [58–65]) nie korzystano z teorii reprezentacji grup.

Skończeniowymiarowe przestrzenie afiniczne „z dodatkowymi strukturami” opisujemy jako odpowiednie „struktury algebraiczne” i opisujemy niektóre ich automorfizmy. Tak otrzymywane grupy automorfizmów są jednocześnie grupami Lie przekształceń, ale ten aspekt, podobnie jak „teoriomnogościowe” sformułowanie teorii grup przekształceń Rychlewskiego [45], jest poza zakresem przedstawionej pracy.