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J. J. Sławianowski, V. Kovalchuk, A. Sławianowska,
B. Gołubowska, A. Martens, E. E. Rożko, Z. J. Zawistowski

INVARIANT GEODETIC SYSTEMS ON LIE GROUPS AND AFFINE MODELS OF INTERNAL AND COLLECTIVE DEGREES OF FREEDOM

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Preface

In some of our earlier papers including rather old ones [21, 25, 26, 27, 28, 46, 47, 48, 49, 57, 58, 62, 63, 65, 66, 67, 68, 73, 75, 77, 79] we have discussed the concept of affinely-rigid body, i.e., continuous, discrete, or simply finite system of material points subject to such constraints that all affine relations between its elements are frozen during any admissible motion. For example, all material straight lines remain straight lines in the course of evolution, and their parallelism is also a constant, non-violated property. Unlike this, the metrical features, like distances and angles, need not be preserved. In other words, such a body is restricted in its behaviour to rigid translations, rigid rotations, and homogeneous deformations. Models of this kind may be successfully applied in a very wide spectrum of physical problems like nuclear dynamics [9] (droplet model of the atomic nuclei), molecular vibrations, macroscopic elasticity [19, 20, 57, 58, 62, 63, 65, 66, 67, 68, 73, 75] (in situations when the length of excited waves is comparable with the size of the body), in the theory of microstructured bodies [66] (micromorphic continua), in geophysics [8, 15] (the theory of the shape of Earth), and even in large-scale astrophysics (vibrating stars, vibrating concentrations of the cosmic substratum, like galaxies or concentrations of the interstellar dust).

From the purely mathematical point of view such a model provides an interesting example of a system with the group-theoretical background of the geometry of degrees of freedom [2, 32, 33, 59, 66, 68, 69]. It is an affine generalization of the usual rigid top with the orthogonal group replaced by the linear one (isometries replaced by affine transformations). Let us remind also that there is an interesting formulation of the general non-constrained continuum mechanics based on the infinite-dimensional "Lie group" of all diffeomorphisms or volume-preserving diffeomorphisms

[1, 2, 7, 17, 18, 42, 43]. This theory is rather complicated (although heuristically very fruitful) because of serious mathematical problems with infinite-dimensional groups. The mechanics of an affinely-rigid body is a simple compromise between rigid-body-mechanics and such a continuum theory, because admitting deformative degrees of freedom it is simultaneously based on the finite-dimensional framework.

Let us stress, however, that, in spite of its non-questionable physical applicability and formally interesting features, the referred mechanics of affinely-rigid body is in a sense disappointing from the point of view of the mathematical theory of Lie group motivated systems. The point is that in the latter theory it is not only kinematics (finally, geometry of degrees of freedom) but also dynamics that is ruled by the underlying group. Due to the isotropy of the physical space, Lagrangian of a free rigid top, i.e., its kinetic energy, is invariant under all left regular translations (all spatial rotations); the same is valid, of course, for the resulting equations of motion (Euler equations). If the material structure of the top is isotropic (spherical inertial tensor), then the model is also invariant under right regular translations. When formulating the theory of ideal incompressible fluids in terms of the group of all volume-preserving diffeomorphisms, one obtains an infinite-dimensional Hamiltonian system invariant under all right regular translations. This is due to the fact that in the usual Euler description of fluid its Lagrangian coordinates are not very essential, and the fluid particles have a rather limited individuality. Summarizing, in these theories one deals with Lagrangians or Hamiltonians based on left-, right-, or even two-side invariant metric tensors on the Lie group used as a configuration space. It is never the case in the above-quoted model of affinely-rigid body. This brings about the question as to the hypothetic affine counterpart of left- and right-invariant geodetic models on the orthogonal group and their potential perturbations. This interest is at least academically motivated. But at the same time, from the physical point of view such models look rather esoteric. In any case, the previously mentioned applications of affine collective modes are dynamically well-established, because they are based on the d'Alembert principle in theory of constrained systems. There are, however, some indications that physical applicability is not a priori excluded.

This problem has to do with the very philosophy of the origin of collective and internal degrees of freedom. We say that a "large" system of material points (continuous, denumerable, or just finite admitted) has collective modes when there exists a "small" number of parameters $q^1, \ldots, q^n$

that are dynamically relevant, i.e., satisfy an approximately autonomous system of evolution equations, if for our purposes the kinematical information about the system, encoded in them, is sufficient, and (very important!) if they depend on individual particles in a non-local way. The latter means that positions and velocities of all particles enter the q^i-variables on essentially equal footing, with the same strength, order of magnitude, so to speak. This is, of course, a rough, qualitative introduction of the term, but there is no place here to develop a rigorous mathematical description. As a mathematical model we can realize some quotient manifolds of multiparticle state spaces or their submanifolds (e.g., representatives of cosets). On the contrary, internal degrees of freedom are described in terms of fibre bundles over the physical space, space-time, or the configuration space. They give an account of phenomena which are either essentially non-extended in space, or perhaps cannot be described in terms of composed systems because their spatial details are unapproachable to our experimental abilities. For example, from the point of view of contemporary science, spin systems seem to be based on essentially internal quantities [37, 84]. In any case, spin media do not look like the Cosserat continuous limit of discrete systems of molecular "gyroscopes". The latter model works successfully in the theory of Van der Waals crystals and granular media.

Apparently, the most natural and intuitive origin of collective modes, e.g., of some microstructure variables, is based on the mechanism of constraints and the d'Alembert principle. Collective motion is then "large", whereas non-collective one is "small" and merely reduced to some vibrations about the appropriate constraint submanifold. The collective kinetic energy, i.e., dynamical metric element, is obtained from the restriction of the total one to the constraints surface (the first fundamental quadratic form). This corresponds to the classical relationship between kinetic energy and inertia [2, 11, 12, 82]. In this case, as a rule, the collective kinetic energy is invariant under a proper subgroup of a group underlying geometry of the constraints submanifold. But one can also realize another mechanism, namely, such one that the hidden non-collective motion is just large, and that the emerging collective modes have to do with the averaged behaviour of hidden modes, i.e., with the time dependence of some relatively slowly-varying mean values. Then it is quite natural to expect that the collective Lagrangian will be based on a kinetic energy whose underlying dynamical metric tensor will be non-interpretable in terms of the restriction of the usual multi-particle metric tensor of the kinetic energy to the constraints

manifold (i.e., to the first fundamental form of constraints). Similarly, equations of motion need not be derivable from the usual d'Alembert principle based on the original spatial metric. Therefore, the relationship between kinetic energy and inertia may become rather non-classical, to some extent exotic in comparison with the usual requirements (cf., e.g., the discussion by Capriz and Trimarco [11, 12, 84]). In such situations the only reasonable procedure is to postulate the kinetic term of the Lagrangian on the basis of some natural and physically justified postulates. Let us mention two examples from the two completely opposite scales of the physical phenomena, namely, the atomic nuclei and vibrating-rotating stars (by the way, the neutron stars are in a sense exotic and gigantic nuclei with $Z = 0$ and enormous A). As objects more close to the Earth one can think, e.g., kinetic bodies as discussed by Capriz, and various non-standard microstructure elements like gas bubbles, voids, and defects in solids [11, 12, 35]. Though bubbles and voids can be hardly treated as constrained pieces of a substance or systems of material points.

Situation is even much more complicated, when one deals with essentially internal degrees of freedom, like, e.g., spin systems [84]. Then, although we have some guiding hints from the theory of extended systems, any choice of Lagrangian, Hamiltonian, or equations of motion is based on some rather hypothetic postulates, first of all, on certain invariance requirements.

There is also another point worth of mentioning. Namely, usually in variational theories of analytical mechanics, Lagrangian consists of the kinetic and potential parts. The first one has to do with inertia, constraints, metric structure, whereas the other one describes true interactions. But even in traditional problems of analytical mechanics there are approaches where the structure of interactions is encoded in an appropriate metric structure, i.e., in a kind of kinetic term. There is a well-known example, namely, the Jacobi-Maupertuis variational principle. If ds is the usual metric (arc) element of the configuration space, and V is the potential energy, then one uses a modified metric [2]

$$d\sigma_V = \sqrt{E - V}\, ds,$$

where E denotes a fixed energy value. This is so-called isoenergetic dynamics, based on the homogeneous "Lagrangian"

$$\mathcal{L} = \sqrt{E - V} \sqrt{g_{ij} \frac{dq^i}{d\lambda} \frac{dq^j}{d\lambda}},$$

λ denoting an arbitrary parameter (not time). This variational principle, based on the metric element $d\sigma_V$, gives trajectories with the energy value E, but without the time-dependence. There are also spatiotemporal forms of this principle, where the time variable occurs as one of coordinates q^i, and there is no restriction to the fixed energy value.

In a slightly different context, in certain problems we will follow the idea of encoding the interaction structure in an appropriately postulated kinetic energy form, i.e., metric tensor on the configuration space.

As mentioned, we concentrate below on models with kinematics (and dynamics) ruled by the linear group $\mathrm{GL}(n,\mathbb{R})$, or, more rigorously, affine group $\mathrm{GAf}(n,\mathbb{R})$ (physically $n = 2, 3$). Of course, the usual rigid body in n dimensions is ruled by $\mathrm{SO}(n,\mathbb{R})$, or, if translations are taken into account, by the isometry group $\mathrm{E}(n,\mathbb{R}) = \mathrm{SO}(n,\mathbb{R}) \times_s \mathbb{R}^n$. But there are also other possibilities of finite-dimensional collective modes, e.g., $\mathrm{SL}(n,\mathbb{R})$ (or $\mathrm{SL}(n,\mathbb{R}) \times_s \mathbb{R}^n$), i.e., incompressible affinely-rigid body, or, just conversely, the Weyl group $\mathbb{R}^+\mathrm{SO}(n,\mathbb{R})$ generated by rotations and translations (the shape of the body is preserved, but not necessarily its size). In some future we are going to investigate systems ruled by the projective group in n dimensions, $\mathrm{Pr}(n,\mathbb{R}) \simeq \mathrm{SL}(n+1,\mathbb{R})$, cf., e.g., [78]. This is quite a natural extension of affinely-rigid body, when the system of material straight-lines is preserved but their parallelism may be violated. Another interesting model would be given by the Euclidean-conformal group $\mathrm{CO}(n,\mathbb{R})$. Let us mention that there was also some very interesting attempt by unjustly forgotten German physicist Westpfahl [85] who invented the idea of using the unitary group U(3) as a basis for collective modes in three dimensions, quite independently of later applications of unitary symmetry in elementary particle physics.

Finally, it is quite often so that the complexification idea leads to physically interesting results. It is not excluded that complexifying the physical space $\mathbb{R}^n$ to $\mathbb{C}^n$ and replacing the real groups $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{U}(n)$ by $\mathrm{GL}(n,\mathbb{C})$ we could obtain some interesting models of collective or internal degrees of freedom [88, 90, 91]. The idea is particularly tempting, because $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{U}(n)$ are two different (and qualitatively opposite) real forms of the same complex group $\mathrm{GL}(n,\mathbb{C})$. But of course, such exotic ideas are rather far from realization, and they are mentioned here only because of their obvious conceptual relationship with the usual and generalized models of affinely-rigid bodies.

The group space is a particular model of systems with kinematics and

dynamics ruled by a Lie group. In general, the microstructure or collective configuration space (the manifold $\mathcal{M}$ in the sense of Capriz book [11] and related papers) is a homogeneous quotient space G/H. Here G is a fundamental group of the model, and H is an appropriate subgroup of G, not necessarily normal one, thus G/H need not inherit the group structure from G [11, 12].

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Contents

Chapter 1

Introduction

Dynamical systems based on Lie groups and their homogeneous spaces are widely used as a model of internal and collective degrees of freedom [11, 12, 13, 14, 40, 41]. They present also interest by themselves from the purely mathematical point of view. They are realistic and quite often they possess rigorous analytical solutions in terms of special functions and power series; this is probably due to the analytical structure of Lie groups. The first step of analysis is the theory of left- and right-invariant geodetic systems, when the Lagrangian and total energy are identical with the kinetic energy expression based on an appropriate Riemannian structure of G.

For simplicity let us use the language of linear groups; by the way, non-linear groups are exceptional in applications, and the most known examples are the universal covering groups $\overline{\mathrm{GL}(n,\mathbb{R})}$, $\overline{\mathrm{SL}(n,\mathbb{R})}$ of the indicated linear groups. For any curve $\mathbb{R} \ni t \mapsto g(t) \in G$ its tangent vectors $\dot{g}(t) \in T_{g(t)}G$ may be transported to the Lie algebra $G' = T_eG$ with the help of right or left $g(t)^{-1}$-translations, resulting in quantities:

$$\Omega(t) := \dot{g}(t)g(t)^{-1}, \qquad \hat{\Omega}(t) := g(t)^{-1}\dot{g}(t).$$

In this way the tangent and cotangent bundles TG, T^*G may be, in two canonical ways, identified with the Cartesian products:

$$TG \simeq G \times G', \qquad T^*G \simeq G \times G'^*.$$

It is clear that the left and right regular translations

$$g \mapsto L_k(g) = kg, \qquad g \mapsto R_k(g) = gk$$

transform quasi-velocities either according to the adjoint rule or trivially:

$$\begin{aligned} L_k \quad &: \qquad \Omega \mapsto \mathrm{Ad}_k \Omega = k\Omega k^{-1}, \qquad \hat{\Omega} \mapsto \hat{\Omega}, \\ R_k \quad &: \qquad \Omega \mapsto \Omega, \quad \hat{\Omega} \mapsto \mathrm{Ad}_{k^{-1}} \hat{\Omega} = k^{-1}\hat{\Omega}k. \end{aligned}$$

Left-invariant geodetic systems on G are based on kinetic energies, which are quadratic forms of $\hat{\Omega}$ with constant coefficients. If G is non-Abelian, then $\hat{\Omega}$ is a non-holonomic quasi-velocity and the corresponding Riemannian structure on G is curved. Similarly, right-invariant kinetic energies are quadratic forms of Ω with constant coefficients. As a canonical example of left-invariant systems we can realize the free rigid body in n dimensions, $G = \mathrm{SO}(n, \mathbb{R})$ (if we neglect translational motion). If the rigid body is spherical (its inertial tensor is completely degenerate), then T is also right-invariant, and the underlying metric tensor on G is proportional to the Killing tensor. Such a pattern may be followed in all semisimple Lie groups [2, 44, 45, 53]. Quite a different example is provided by the theory of the ideal fluids [2]. The configuration space is identified with SDiff $\mathbb{R}^3$ — the infinite-dimensional group of all volume-preserving diffeomorphisms of $\mathbb{R}^3$ (provided that we discuss the physical three-dimensional case). If we admitted the fluid to be compressible, we would have to use the full group Diff $\mathbb{R}^3$ of all diffeomorphisms. The functional of kinetic energy is invariant under right regular translations in SDiff $\mathbb{R}^3$. What concerns left regular translations, it is invariant only under the six-dimensional isometry group of $\mathbb{R}^3$. The reason for this relatively poor left-hand-side invariance is that the kinetic energy expression depends in an essential way on the spatial metric tensor. At the same time, from the point of view of the material space, the particles of fluid have a rather limited individuality, and that is why the kinetic energy form of incompressible fluid is invariant under the huge group of sufficiently smooth and volume-preserving "permutations" of particles, i.e., under SDiff $\mathbb{R}^3$.

In some of our earlier papers [24, 25, 26, 27, 28, 46, 47, 48, 49, 57, 58, 62, 63, 65, 66, 67, 68, 73, 75, 77, 79] we discussed the object called "affinely-rigid body", i.e., the system of material points "rigid" in the sense of affine geometry, i.e., all affine relationship between constituents being kept fixed during any admissible motion. Such a model is geometrically interesting in itself and has a wide range of applications in macroscopic elasticity, mechanics of micromorphic continua with internal degrees of freedom, molecular vibrations, nuclear dynamics, vibrations of astrophysical objects, and the

theory of the shape of Earth [8, 15]. Analytically, the configuration space of n-dimensional affinely-rigid body may be identified with the semi-direct product $\mathrm{GL}(n,\mathbb{R}) \times_s \mathbb{R}^n$, or simply $\mathrm{GL}(n,\mathbb{R})$ when we neglect translational degrees of freedom.

The kinetic energy of an extended affinely-rigid body in Euclidean space may be calculated in the usual way, by summation of kinetic energies of its constituents. Velocity vectors are squared with the use of the fixed metric tensor of the physical space. The resulting metric tensor of the configuration space is flat, and it is not invariant either under left or right regular translations, except two subgroups isomorphic with the Euclidean group $\mathrm{SO}(n,\mathbb{R}) \times_s \mathbb{R}^n$. Because of this the resulting geodetic model, although kinematically based on the group manifold, dynamically is incompatible with it. Besides, it is physically non-realistic and useless, because geodetics are straight lines in $\mathrm{L}(n,\mathbb{R}) \times_s \mathbb{R}^n$, therefore, in certain directions the body would suffer a non-limited extension or squeezing. It is impossible to avoid such non-physical catastrophic phenomena without introducing some potential term.

The very taste and mathematical machinery of systems with group-manifold degrees of freedom consist in the invariance of geodetic models under the total group of regular translations. This motivates the search for left- or right-invariant kinetic energies, i.e., Riemannian structures on $\mathrm{GL}(n,\mathbb{R}) \times_s \mathbb{R}^n$ or $\mathrm{GL}(n,\mathbb{R})$. The first step is purely mathematical: the very construction and some primary analysis. Later on some hypotheses are formulated concerning the physical applicability of such apparently exotic "non-d'Alembertian" models.

Chapter 2

Kinematics and Poisson brackets

2.1 Basic ideas

Let us remind briefly the basic ideas concerning the extended affinely-rigid body in a flat Euclidean space [57, 58, 62, 63, 65, 66, 67, 68, 73, 75]. It is convenient to use the standard terms of continuum mechanics, although the model applies also to discrete or finite systems of material points (provided there exist at least $n+1$ material points in n-dimensional space). Two Euclidean spaces are used, namely, the material space (N, U, η) and the physical space (M, V, g); the symbols N, M denote the underlying sets, U and V are their linear spaces of translations, and $\eta \in U^* \otimes U^*$, $g \in V^* \otimes V^*$ are metric tensors. We put $\dim N = \dim M = n$. The points of N are labels of material points. The configuration space Q of affinely-rigid body in M is given by $\mathrm{AfI}(N, M)$, i.e., the manifold of affine isomorphisms of N onto M. Obviously, it is an open submanifold of $\mathrm{Af}(N, M)$ — the affine space of all affine mappings of N into M (including non-invertible ones). In some configuration $\Phi \in Q$ the material point $a \in N$ occupies the spatial position $x = \Phi(a) \in M$. The co-moving, i.e., Lagrangian, mass distribution within the body will be described by the constant (time-independent) positive measure μ on N; it may be δ-like (concentrated at single points), continuous with respect to the Lebesgue measure, or mixed. Cartesian (La-

grange) coordinates a^K in N are chosen in such a way that their origin is placed at the centre of mass $\mathcal{C}$, i.e.,

$$\int a^K d\mu(a) = 0.$$

The manifold $\mathrm{AfI}(N, M)$ may be identified with the Cartesian product $M \times \mathrm{LI}(U, V)$, where $\mathrm{LI}(U, V)$ denotes the manifold of all linear isomorphism of U onto V; it is an open submanifold of the linear space $\mathrm{L}(U, V)$. The first factor refers to translational motion, i.e., to the centre of mass position $x = \Phi(\mathcal{C})$. The linear part of Φ, $\varphi = L[\Phi] = D\Phi \in \mathrm{LI}(U, V)$, describes the relative (internal) motion. Analytically, when some Cartesian coordinates in M are used, motion is described by the dependence of Euler (current) coordinates on Lagrangian (material) ones and on the time variable:

$$\Phi(t, a)^i = \varphi^i{}_K(t) a^K + x^i(t).$$

In practical calculations it is often technically convenient, although may be geometrically misleading, to identify both U and V with $\mathbb{R}^n$ and Q with semi-direct product $\mathrm{GAf}(n, \mathbb{R}) \simeq \mathrm{GL}(n, \mathbb{R}) \times_s \mathbb{R}^n$. Another natural model of Q is $M \times \mathrm{F}(V)$, where $\mathrm{F}(V)$ denotes the manifold of all linear frames in V. By the way, $\mathrm{F}(V)$ as a model of internal (relative-motion) degrees of freedom is essentially identical with $\mathrm{LI}(U, V)$ if we put $U = \mathbb{R}^n$ and use the natural isomorphism between linear mappings $\varphi \in \mathrm{LI}(\mathbb{R}^n, V)$ and co-moving frames $e \in \mathrm{F}(V)$ frozen into the body and attached at the centre of mass. This must be done when the body is infinitesimal and the relative motion is replaced by the dynamics of essentially internal degrees of freedom. Then $\mathbb{R}^n$ becomes the micromaterial space of internal motion.

Inertia of the body is described by two constant quantities, namely, the total mass

$$m := \int_N d\mu(a)$$

and the second-order moment of internal inertia $J \in U \otimes U$,

$$J^{KL} := \int_N a^K a^L d\mu(a)$$

(cf. [57, 58, 62, 63, 65, 66, 67, 68, 73, 75]); it is symmetric and positively-definite.

Summing up the kinetic energies of constituents,

$$T = \frac{1}{2} g_{ij} \int \frac{\partial \Phi^i}{\partial t}(t,a) \frac{\partial \Phi^j}{\partial t}(t,a) d\mu(a),$$

one obtains:

$$T = T_{\rm tr} + T_{\rm int} = \frac{m}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^i}{dt} + \frac{1}{2} g_{ij} \frac{d\varphi^i{}_A}{dt} \frac{d\varphi^i{}_B}{dt} J^{AB}; \qquad (2.1)$$

the symbols "tr" and "int" refer, obviously, to the translational and internal (relative) terms.

The phase space of our system may be identified with the manifold

$$P := M \times {\rm LI}(U,V) \times V^* \times {\rm L}(V,U).$$

The factor V^* refers to translational canonical momentum, whereas ${\rm L}(V,U)$ to the internal one, in the sense of the obvious pairing between $\pi \in {\rm L}(V,U)$ and generalized internal velocity $\xi \in {\rm L}(U,V)$:

$$< \pi, \xi >= {\rm Tr}(\pi \cdot \xi) = {\rm Tr}(\xi \cdot \pi).$$

Cartesian coordinates in M generate parametrization p_i, $p^A{}_i$ of canonical momenta. For Lagrangians of the form $L = T - V(x,\varphi)$ Legendre transformation

$$p_i = m g_{ij} \frac{dx^j}{dt}, \qquad p^A{}_i = g_{ij} \frac{d\varphi^j{}_B}{dt} J^{BA} \qquad (2.2)$$

leads to the following kinetic term of the Hamiltonian:

$$\mathcal{T} = \frac{1}{2m} g^{ij} p_i p_j + \frac{1}{2} g^{ij} p^A{}_i p^B{}_j \widetilde{J}_{AB},$$

where, obviously, g^{ij} are components of the reciprocal contravariant metric of g, and $\widetilde{J} \in U^* \otimes U^*$ is reciprocal to J, $\widetilde{J}_{AC} J^{CB} = \delta_A{}^B$, do not confuse it with J with the η-lowered indices. This kinetic term (and its underlying flat metric on Q) is invariant under Abelian additive translations in $Q = M \times {\rm LI}(U,V)$; those in the second term are meant in the sense

$${\rm LI}(U,V) \ni \varphi \mapsto \varphi + \alpha, \qquad \alpha \in {\rm L}(U,V). \qquad (2.3)$$

Therefore, without the interaction term (for $L = T$), the Hamiltonian generators p_i, $p^A{}_i$ are constants of motion. However, as mentioned above, such

geodetic models for deformable bodies are physically non-interesting, because they predict unlimited expansion, contraction, and passing through singular configurations with $\det \varphi = 0$. The latter, although non-acceptable in continuum mechanics, may be to some extent admissible in mechanics of discrete bodies. If we once decide that the internal configuration space is given by $\mathrm{LI}(U, V)$, then the above transformation group is only local. At the same time, even for purely geodetic systems, as mentioned, there is no invariance under geometrically interesting affine groups of left or right affine regular translations in Q. Even if, at the present stage, models with affinely-invariant kinetic energy might seem rather academic, they present some interest at least from the purely mathematical point of view. Besides, some physical applications seem to be possible in hydrodynamics, astrophysics, nuclear dynamics, and in certain elastic problems. It is interesting that even without any genuine interactions, on the purely geodetic level such models may predict bounded and stable elastic vibrations of incompressible bodies. It is so as if the interaction was encoded in the very kinetic energy, i.e., configuration metric, so as it is, e.g., in Jacobi-Maupertuis variational principle. To formulate such models we must introduce and partially remind certain geometric objects.

2.2 Introducing geometric objects

Affine velocity in laboratory representation, i.e., expressed in terms of space-fixed frames, is defined as

$$\Omega := \frac{d\varphi}{dt}\varphi^{-1} \in \mathrm{L}(V), \qquad \Omega^i{}_j = \frac{d\varphi^i{}_K}{dt}\left(\varphi^{-1}\right)^K{}_j.$$

The corresponding co-moving object, related to the body-fixed frame, is given by

$$\hat{\Omega} := \varphi^{-1}\frac{d\varphi}{dt} \in \mathrm{L}(U), \qquad \hat{\Omega}^A{}_B = \left(\varphi^{-1}\right)^A{}_i \frac{d\varphi^i{}_B}{dt}.$$

Obviously,

$$\Omega = \varphi\hat{\Omega}\varphi^{-1}, \qquad \Omega^i{}_j = \varphi^i{}_A\hat{\Omega}^A{}_B\left(\varphi^{-1}\right)^B{}_j.$$

These are Lie-algebraic objects corresponding to the structure of Q as the group space of a Lie group. They provide an affine counterpart of the rigid-body angular velocities, and in fact reduce to them when φ is confined to

the manifold of isometries of (U, η) onto (V, g); then they become skew-symmetric respectively with respect to η or g.

The object Ω may be represented in terms of continua as a gradient of the Euler velocity field, namely, the material point passing the fixed spatial point y has the translational velocity:

$$^{E}v(y)^{i} = \frac{dx^{i}}{dt} + \Omega^{i}{}_{j}(y^{j} - x^{j}),$$

i.e., simply $\Omega^{i}{}_{j}y^{j}$ in the instantaneous rest frame of the centre of mass, placed also at the instantaneous position of this centre in M. Similarly, $d\varphi^{i}{}_{B}/dt$ has to do with the gradient of the Lagrange velocity field, because the instantaneous velocity of the a-th particle ($a \in N$) is given by

$$^{L}v(a)^{i} = \frac{dx^{i}}{dt} + \frac{d\varphi^{i}{}_{K}}{dt}a^{K}$$

(concerning the standard concepts of continuum mechanics consult, e.g., [19, 20, 36]).

In certain problems it is also convenient to express the centre of mass translational velocity $v^{i} = dx^{i}/dt$ in co-moving terms,

$$\hat{v}^{A} = \left(\varphi^{-1}\right)^{A}{}_{i}v^{i}.$$

It is very convenient to introduce the canonical affine spin, also in two representations, the spatial and co-moving ones $\Sigma \in \mathrm{L}(V)$, $\hat{\Sigma} \in \mathrm{L}(U)$. In terms of coordinates they are given by the following formulas:

$$\Sigma^{i}{}_{j} = \varphi^{i}{}_{A}p^{A}{}_{j}, \qquad \hat{\Sigma}^{A}{}_{B} = p^{A}{}_{i}\varphi^{i}{}_{B}.$$

As previously,

$$\Sigma = \varphi\hat{\Sigma}\varphi^{-1}, \qquad \Sigma^{i}{}_{j} = \varphi^{i}{}_{A}\hat{\Sigma}^{A}{}_{B}\left(\varphi^{-1}\right)^{B}{}_{j}.$$

They are purely Hamiltonian quantities defined on the phase space; without any precisely defined Lagrangian or Hamiltonian we cannot relate them to generalized velocities. It is seen, however, that they are dual objects to affine velocities, i.e., they are non-holonomic canonical momenta conjugate to them in the sense of following pairing:

$$< \Sigma, \Omega > = < \hat{\Sigma}, \hat{\Omega} > := \mathrm{Tr}(\Sigma\Omega) = \mathrm{Tr}(\hat{\Sigma}\hat{\Omega}) = p^{A}{}_{i}v^{i}{}_{A},$$

where $v^i{}_A$ are generalized velocities of internal (relative) motion. This canonical isomorphism between Lie algebras $\mathrm{GL}(U)' = \mathrm{L}(U)$, $\mathrm{GL}(V)' = \mathrm{L}(V)$ and their duals simplifies remarkably all formulas and considerations.

It is clear that quantities $\Sigma^i{}_j$ are Hamiltonian generators of $\mathrm{GL}(V)$ acting on $\mathrm{LI}(U,V)$ through the left translations:

$$\varphi \mapsto A\varphi, \qquad \varphi \in \mathrm{LI}(U,V), \quad A \in \mathrm{GL}(V). \tag{2.4}$$

Similarly, $\hat{\Sigma}^A{}_B$ generate right regular translations in the internal configuration space:

$$\varphi \mapsto \varphi B, \qquad \varphi \in \mathrm{LI}(U,V), \quad B \in \mathrm{GL}(U). \tag{2.5}$$

In continuum mechanics these mappings are referred to, respectively, as spatial and material transformations; in this case they include rotations and homogeneous deformations. Obviously, to use correctly such terms we must be given metric tensors in V and U. Then the g-antisymmetric part of Σ and the η-antisymmetric part of $\hat{\Sigma}$ generate, respectively, spatial and material rigid rotations; the symmetric parts generate deformations.

The doubled antisymmetric parts are referred to as spin S and vorticity V [16],

$$S^i{}_j = \Sigma^i{}_j - g^{ik}g_{jm}\Sigma^m{}_k, \qquad V^A{}_B = \hat{\Sigma}^A{}_B - \eta^{AC}\eta_{BD}\hat{\Sigma}^D{}_C. \tag{2.6}$$

Attention! There is an easy mistake possibility: if motion is not metrically-rigid, then V is not a co-moving representation of S, i.e.,

$$S^i{}_j \neq \varphi^i{}_A V^A{}_B \left(\varphi^{-1}\right)^B{}_j.$$

Just as translational velocity, the canonical linear momentum may be expressed in co-moving terms according to the following rule:

$$\hat{p}_A = p_i \varphi^i{}_A.$$

The objects Ω and Σ are invariant under material transformations, but the spatial action of $A \in \mathrm{GL}(V)$ transforms them according to the adjoint rule, i.e.,

$$\Omega \mapsto A\Omega A^{-1}, \qquad \Sigma \mapsto A\Sigma A^{-1}. \tag{2.7}$$

On the contrary, $\hat{\Omega}$ and $\hat{\Sigma}$ are invariant under $\mathrm{GL}(V)$ but experience the inverse adjoint rule under $B \in \mathrm{GL}(U)$, i.e.,

$$\hat{\Omega} \mapsto B^{-1}\hat{\Omega}B, \qquad \hat{\Sigma} \mapsto B^{-1}\hat{\Sigma}B.$$

This formally agrees with formulas for systems with configuration spaces identical with Lie groups, but one must stress that there are some subtle differences due to the fact that $\mathrm{LI}(U,V)$ is not a Lie group (may be identified with it, but there is an infinity mutually equivalent identifications).

The translational or orbital affine momentum with respect to some point $\mathcal{O} \in M$ is defined as follows:

$$\Lambda(\mathcal{O})^i{}_j := x^i p_j,$$

where x^i are Cartesian coordinates of the $\mathcal{O}$-radius vector of the current position of the centre of mass in M. The total affine momentum with respect to $\mathcal{O}$ is given by

$$I(\mathcal{O})^i{}_j := \Lambda(\mathcal{O})^i{}_j + \Sigma^i{}_j.$$

$\Lambda(\mathcal{O})$ and $I(\mathcal{O})$ depend explicitly on the choice of $\mathcal{O}$. Unlike this, Σ is objective (in a fixed Galilean reference frame). There is a complete analogy with the properties of angular momentum, the doubled g-antisymmetric part of the above objects. The quantity $I(\mathcal{O})$ is a Hamiltonian generator of the group of affine transformations of M preserving $\mathcal{O}$ ($\mathcal{O}$-centred affine subgroup).

2.3 Basic Poisson brackets

Poisson brackets of Σ-quantities follow directly from the standard ones for x^i, p_i, $\varphi^i{}_A$, $p^A{}_i$. The non-vanishing ones are simply given by the structure constants of linear groups,

$$\{\Sigma^i{}_j, \Sigma^k{}_l\} = \delta^i{}_l \Sigma^k{}_j - \delta^k{}_j \Sigma^i{}_l,$$

similarly for Λ, I, and

$$\begin{aligned} \{\hat{\Sigma}^A{}_B, \hat{\Sigma}^C{}_D\} &= \delta^C{}_B \hat{\Sigma}^A{}_D - \delta^A{}_D \hat{\Sigma}^C{}_B, \\ \{\Sigma^i{}_j, \hat{\Sigma}^A{}_B\} &= 0. \end{aligned}$$

There are also non-vanishing Poisson brackets related to the left or right affine groups GAf(M), GAf(N). Here belong the above ones and besides, those involving linear momenta,

$$\begin{aligned} \{\hat{\Sigma}^A{}_B, \hat{p}_C\} &= \delta^A{}_C \hat{p}_B, \\ \{I^i{}_j, p_k\} = \{\Lambda^i{}_j, p_k\} &= \delta^i{}_k p_j. \end{aligned}$$

If F is any function depending only on the configurations variables, then, obviously,

$$\begin{aligned} \{F, \Sigma^i{}_j\} &= \varphi^i{}_A \frac{\partial F}{\partial \varphi^j{}_A}, \\ \{F, \Lambda^i{}_j\} &= x^i \frac{\partial F}{\partial x^j}, \\ \{F, \hat{\Sigma}^A{}_B\} &= \varphi^i{}_B \frac{\partial F}{\partial \varphi^i{}_A}. \end{aligned}$$

Geometric meaning of the last formulas is clear, because the differential operators used on their right-hand sides are identical with vector fields on Q generating the action of one-parameter subgroups of GAf(M) and GAf(N).

As mentioned, the above Poisson brackets follow directly from the standard definition [2, 23, 29]

$$\{F, G\} := \frac{\partial F}{\partial q^\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q^\alpha},$$

where q^α are generalized coordinates and p_α are their conjugate canonical momenta. In our model q^α are given by x^i, $\varphi^i{}_A$, and p_α by p_i, $p^A{}_i$. In applications it is sufficient to remember that

$$\{q^\alpha, q^\beta\} = 0, \qquad \{p_\alpha, p_\beta\} = 0, \qquad \{q^\alpha, p_\beta\} = \delta^\alpha{}_\beta,$$

that Poisson bracket is bilinear (over constant reals $\mathbb{R}$), skew-symmetric, i.e., $\{F, G\} = -\{G, F\}$, that it satisfies the Jacobi identity

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0,$$

and finally that

$$\{F, H(G_1, \ldots, G_k)\} = \sum_{p=1}^{k} H_{,p}(G_1, \ldots, G_k)\{F, G_p\},$$

where commas before indices denote the partial derivatives. The formerly-quoted Poisson brackets together with the above rules are sufficient for all calculations concerning equations of motion and their analysis.

2.4 Hamiltonian dynamical models

To define a non-dissipative (Hamiltonian) dynamical model, we must be given some Lagrangian $L(q, \dot{q})$, perform the Legendre transformation, $p_\alpha = \partial L/\partial \dot{q}^\alpha$, invert it, i.e., solve with respect to generalized velocities $\dot{q}^\alpha$, and substitute the result to the energy function $E = \dot{q}^\alpha \partial L/\partial \dot{q}^\alpha - L$. In this way one obtains the Hamilton function $H(q, p)$. Equations of motion may be then expressed in terms of Poisson brackets,

$$\frac{dF}{dt} = \{F, H\},$$

where F runs over some finite family of basic functions, e.g., it can be $(p_i, \Sigma^i{}_j, x^i, \varphi^i{}_A)$, $(\hat{p}_A, \hat{\Sigma}^A{}_B, x^i, \varphi^i{}_A)$, or something else. The basic dynamical laws are given by the balance equations for the linear momentum and affine spin either in laboratory or co-moving representation (one could use equivalently the linear momentum and the total affine momentum, however, the previous choice is more convenient). The procedure based on Poisson brackets and canonical formalism is very often more easy and computationally less embarrassing than the one directly using the Euler-Lagrange equations.

Remark: Legendre transformation may be also expressed in terms of non-holonomic objects, moreover, this is often more convenient and effective than the use of generalized velocities. Expressing Lagrangian in terms of $(v^i, \Omega^i{}_j)$ or $(\hat{v}^A, \hat{\Omega}^A{}_B)$ instead of $(\dot{x}^i, \dot{\varphi}^i{}_A)$, we can describe the Legendre transformation as follows:

$$p_i = \frac{\partial L}{\partial v^i}, \qquad \Sigma^i{}_j = \frac{\partial L}{\partial \Omega^j{}_i},$$

or

$$\hat{p}_A = \frac{\partial L}{\partial \hat{v}^A}, \qquad \hat{\Sigma}^A{}_B = \frac{\partial L}{\partial \hat{\Omega}^B{}_A}.$$

When dealing with the Hamiltonian form of equations of motion, we need often Poisson brackets involving deformation tensors and certain by-products of the inertial tensor, like, e.g., the Eulerian quadrupole of the mass distribution.

2.5 Deformation tensors

Obviously, for systems with affine degrees of freedom the Green and Cauchy deformation tensors $G \in U^* \otimes U^*$, $C \in V^* \otimes V^*$ are respectively given by the following expressions:

$$G = \varphi^* \cdot g, \qquad C = \left(\varphi^{-1}\right)^* \cdot \eta,$$

i.e., in analytical terms:

$$G_{AB} = g_{ij}\varphi^i{}_A\varphi^j{}_B, \qquad C_{ij} = \eta_{AB}\left(\varphi^{-1}\right)^A{}_i\left(\varphi^{-1}\right)^B{}_j.$$

Their inverses $\widetilde{G} \in U \otimes U$, $\widetilde{C} \in V \otimes V$ are defined by:

$$\widetilde{G}^{AC}G_{CB} = \delta^A{}_B, \qquad \widetilde{C}^{ik}C_{kj} = \delta^i{}_j,$$

and one must be careful to avoid mistaking $\widetilde{G}^{AB}$, $\widetilde{C}^{ij}$ with

$$\eta^{AC}\eta^{BD}G_{CD}, \qquad g^{ik}g^{jl}C_{kl}.$$

Therefore, the usual convention of the upper- and lower-case indices may be misleading. Analytically,

$$\widetilde{G}^{AB} = \left(\varphi^{-1}\right)^A{}_i\left(\varphi^{-1}\right)^B{}_j g^{ij}, \qquad \widetilde{C}^{ij} = \varphi^i{}_A\varphi^j{}_B\eta^{AB}.$$

When there is no deformation, i.e., $\varphi \in \mathrm{LI}(U, \eta; V, g)$, then $G = \eta$, $C = g$. The corresponding deformation measures vanishing in the non-deformed state, i.e., Lagrange and Cauchy deformation tensors $E \in U^* \otimes U^*$, $e \in V^* \otimes V^*$ are given by (see, e.g., [19, 20]):

$$E := \frac{1}{2}(G - \eta), \qquad e := \frac{1}{2}(g - C).$$

One uses also their contravariant versions E^{AB}, e^{ij}; unlike $\widetilde{G}^{AB}$, $\widetilde{C}^{ij}$ they are defined via the η- and g-raising of indices.

Remark: G is independent of η and may be defined even if the material space is purely affine, amorphous. Similarly, C is independent of g and is well-defined even if the physical space is metric-free. Therefore, the literally meant term "deformation" is better expressed by E, e than G, C. However, in many formulas G, C are more natural and convenient. Deformation tensors behave under the action of isometries in a very peculiar way, namely, for any $A \in \mathrm{O}(V, g)$, $B \in \mathrm{O}(U, \eta)$, we have:

$$G[A\varphi]_{KL} = G[\varphi]_{KL}, \qquad C[\varphi B]_{ij} = C[\varphi]_{ij},$$

$$G[\varphi B]_{KL} = G[\varphi]_{CD} B^C{}_K B^D{}_L, \qquad C[A\varphi]_{ij} = C[\varphi]_{ab} \left(A^{-1}\right)^a{}_i \left(A^{-1}\right)^b{}_j.$$

By the way, the last two formulas are valid for any $A \in \mathrm{GL}(V)$, $B \in \mathrm{GL}(U)$. The first two equations (invariance rules) imply the following Poisson-bracket rules:

$$\{G_{KL}, S^i{}_j\} = 0, \qquad \{C_{ij}, V^A{}_B\} = 0,$$

and similarly for E_{KL}, e_{ij}.

Deformation invariants are important mechanical quantities. They are scalar measures of deformation, basic stretchings, which do not contain any information concerning the orientation of deformation (its principal axes) in the physical or material space. They may be chosen in various ways, but in an n-dimensional space exactly n of them may be functionally independent. The particular choice of n basic invariants depends on the considered problem and on the computational details. When non-specified, they will be denoted by $\mathcal{K}_a$, $a = \overline{1, n}$. Let us define mixed tensors

$$\hat{G} \in U \otimes U^*, \qquad \hat{C} \in V \otimes V^*, \qquad \hat{E} \in U \otimes U^*, \qquad \hat{e} \in V \otimes V^*,$$

namely,

$$\hat{G}^A{}_B := \eta^{AC} G_{CB}, \quad \hat{C}^i{}_j := g^{ik} C_{kj}, \quad \hat{E}^A{}_B := \eta^{AC} E_{CB}, \quad \hat{e}^i{}_j := g^{ik} e_{kj}.$$

A class of possible and geometrically natural choices of $\mathcal{K}_a$ is given by the following expressions:

$$\mathrm{Tr}(\hat{G}^k), \qquad \mathrm{Tr}(\hat{C}^k), \qquad \mathrm{Tr}(\hat{E}^k), \qquad \mathrm{Tr}(\hat{e}^k), \quad k = \overline{1, n}.$$

In certain problems it is convenient to use the following eigenequations:

$$\det[\hat{G}^A{}_B - \lambda \delta^A{}_B] = 0, \qquad \det[\hat{C}^i{}_j - \lambda \delta^i{}_j] = 0,$$

$$\det[\hat{E}^A{}_B - \lambda\delta^A{}_B] = 0, \qquad \det[\hat{e}^i{}_j - \lambda\delta^i{}_j] = 0.$$

These are n-th order algebraic (polynomial) equations with respect to λ. Their solutions provide one of possible choices of basic invariants. Another, very convenient one is given by coefficients at λ^p, $p = \overline{0,(n-1)}$ [19, 20] (the coefficient at λ^n is standard and equals one). Deformation invariants are non-sensitive with respect to spatial and material isometries, i.e., for any $A \in \mathrm{O}(V,g)$, $B \in \mathrm{O}(U,\eta)$ we have:

$$\mathcal{K}_a[A\varphi B] = \mathcal{K}_a[\varphi].$$

This implies the obvious Poisson brackets:

$$\{\mathcal{K}_a, S^i{}_j\} = \{\mathcal{K}_a, V^A{}_B\} = 0.$$

In certain computational problems, but also in theoretical analysis, it is very convenient to use quantities $Q^a = \sqrt{\lambda_a}$, where λ_a are solutions of the above eigenequations, or $q^a = \ln Q^a$ (i.e., $Q^a = \exp(q^a)$). The eigenvalues of $\hat{C}$ equal $(\lambda_a)^{-1} = (Q^a)^{-2} = \exp(-2q^a)$.

Any function F on the configuration space which depends on φ only through the deformation invariants is doubly isotropic, i.e., it satisfies the condition $F(A\varphi B) = F(\varphi)$ for any $A \in \mathrm{O}(V,g)$, $B \in \mathrm{O}(U,\eta)$, and $\varphi \in \mathrm{LI}(U,\eta;V,g)$. All such functions have vanishing Poisson brackets with spin and vorticity, i.e.,

$$\{F, S^i{}_j\} = \{F, V^A{}_B\} = 0.$$

In certain formulas we need the spatial inertial quadrupole,

$$J[\varphi]^{ab} = \varphi^a{}_K \varphi^b{}_L J^{KL}.$$

It is related to J^{KL} just as $\widetilde{C}$ is to η. When the body is inertially isotropic, $J[\varphi]$ becomes proportional to the inverse Cauchy deformation tensor. Unlike the co-moving internal tensor $J \in U \otimes U$, $J[\varphi] \in V \otimes V$ is configuration-dependent, thus variable in time.

Chapter 3

Traditional d'Alembert model

3.1 Dynamical equations of motion

At least for the comparison with more exotic (although geometrically and perhaps physically interesting) suggestions we must start with a brief reporting and extension of the traditional model based on the d'Alembert principle. As shown in [60, 61, 62, 63, 64, 65], Lagrangians of the form $L = T - V(x, \varphi)$ with T given by (2.1) lead to the following dynamical laws:

$$\begin{aligned} \frac{dp_i}{dt} &= -\frac{\partial V}{\partial x^i} = Q_i, \\ \frac{d\Sigma^i{}_j}{dt} &= \Omega^i{}_m \Sigma^m{}_j - \varphi^i{}_A \frac{\partial V}{\partial \varphi^j{}_A} = \Omega^i{}_m \Sigma^m{}_j + Q^i{}_j, \end{aligned} \tag{3.1}$$

expressed in terms of Cartesian coordinate systems. This is the balance for fundamental Hamiltonian generators. It becomes a closed dynamical system when considered together with the Legendre transformation (2.2) or its equivalent description

$$p_i = m g_{ij} \frac{dx^j}{dt}, \qquad \Sigma^i{}_j = g_{jk} \Omega^k{}_m J[\varphi]^{mi}. \tag{3.2}$$

Substituting these expressions to the dynamical balance for $p_i, \Sigma^i{}_j$ one obtains some reformulation of the Euler-Lagrange equations. Similarly, some form of canonical Hamilton equations is obtained when the balance (3.1) is unified with the inverse Legendre transformation, i.e.,

$$\frac{dx^i}{dt} = \frac{1}{m} g^{ij} p_j, \qquad \Omega^i{}_j = \widetilde{J}[\varphi]_{jk} \Sigma^k{}_m g^{mi},$$

where, obviously,

$$J[\varphi]^{ik} \widetilde{J}[\varphi]_{kj} = \delta^i{}_j$$

(do not confuse $\widetilde{J}[\varphi]$ with g-shift of indices of $J[\varphi]$).

Obviously, the general balance form may be used for dissipative non-Largangian models. Simply the covariant force Q_i and the generalized internal force $Q^i{}_j$ (affine moment of forces, hyperforce) must involve appropriately defined dissipative forces (in the case of affinely-constrained continuum one can also consider the mutual coupling of mechanical phenomena with discretized thermal effects).

As shown in the mentioned papers, the above equations of motion may be formulated in various equivalent forms adapted to the kind of considered problems. For example, instead of the canonical (Hamilton) form, one can write them down in purely kinematical velocity-based terms, i.e.,

$$m \frac{d^2 x^i}{dt^2} = F^i, \qquad \varphi^i{}_A \frac{d^2 \varphi^j{}_B}{dt^2} J^{AB} = N^{ij}, \tag{3.3}$$

where contravariant forces F^i and hyperforce N^{ij} (affine dynamical moment) may depend on all possible arguments, i.e., t, x^i, $\varphi^i{}_A$, dx^i/dt, and $d\varphi^i{}_A/dt$. Obviously, for potential models they depend only on generalized coordinates and possibly on the time variable t itself, and then

$$F^i = g^{ij} Q_j = -g^{ij} \frac{dV}{dx^j}, \qquad N^{ij} = Q^i{}_k g^{kj} = -\varphi^i{}_A \frac{\partial V}{\partial \varphi^k{}_A} g^{kj}. \tag{3.4}$$

Remark: In spite of the tradition based on Riemannian geometry and relativity theory we shall refrain from the graphical identification of symbols F^i, N^{ij} respectively with Q^i, Q^{ij}. In our treatment this would be just confusing, because we shall use various prescriptions for shifting the tensorial indices, i.e., various isomorphisms between contravariant and covariant objects.

3.2 Monopole and dipole moments

As mentioned, the above equations of the motion (3.3) may be derived directly in Newtonian terms, basing merely on the d'Alembert principle and its underlying spatial metric g in M. The primary quantities of this approach are the monopole and dipole moments of the distributions of linear kinematical momentum and forces within the body. These quantities, just as all high-order multipoles may be defined for any unconstrained system of material points, does not matter finite, discrete, or continuous. Let $\Phi(t,a)^i$ denote as previously Cartesian coordinates of the current position of the a-th material point at the time instant t, $x^i(t)$ be the current position of the centre of mass, and $\mathcal{F}^i(t,x,\Phi(t,a),dx/dt,(\partial\Phi/\partial t)(t,a);a)$ be the density of forces per unit mass. As mentioned, the affine constraints are not yet assumed.

The monopoles are simply the total quantities: the total kinematical momentum k^i (do not confuse it at this stage with the canonical one p_i) and the total force F^i affecting the centre of mass motion, i.e.,

$$\begin{aligned} k^i &= \int \frac{\partial \Phi^i}{\partial t}(t,a) d\mu(a), \\ F^i &= \int \mathcal{F}^i \left(t, x, \Phi(t,a), \frac{dx}{dt}, \frac{\partial \Phi}{\partial t}(t,a); a \right) d\mu(a). \end{aligned}$$

The dipole moments with respect to the centre of mass current position are referred to as kinematical affine spin K^{ij} (do not confuse it at this stage with the canonical one $\Sigma^i{}_j$) and the affine moment of forces N^{ij} (not to be confused with its potential version $Q^i{}_j$). They are given respectively by the following expressions:

$$\begin{aligned} K^{ij} &= \int \left(\Phi^i(t,a) - x^i \right) \left(\frac{\partial \Phi^j}{\partial t}(t,a) - \frac{dx^j}{dt} \right) d\mu(a), \\ N^{ij} &= \int \left(\Phi^i(t,a) - x^i \right) \mathcal{F}^j \left(t, x, \Phi(t,a), \frac{dx}{dt}, \frac{\partial \Phi}{\partial t}(t,a); a \right) d\mu(a). \end{aligned}$$

The dipoles may be also referred to some space-fixed centre $\mathcal{O} \in M$, e.g., the origin of Cartesian coordinates in M. The difference is that "the lever arm" $(\Phi^i - x^i)$ is then replaced by Φ^i itself, and its velocity $\left(\partial\Phi^i/\partial t - dx^i/dt\right)$ by $(\partial\Phi^i/\partial t)(t,a)$. The resulting dipoles will be denoted respectively by $K(\mathcal{O})^{ij}$, $N(\mathcal{O})^{ij}$. For the sake of uniformity, it may be also

convenient to denote the previous dipoles by $K(\mathrm{cm})^{ij}$ and $N(\mathrm{cm})^{ij}$ instead of K^{ij}, N^{ij}. We shall also use affine moments of the centre of mass characteristics with respect to the origin $\mathcal{O}$. Thus, the translational (orbital) affine moment of kinematical linear momentum is as follows:

$$K_{\mathrm{tr}}(\mathcal{O})^{ij} = x^i k^j = m x^i \frac{dx^j}{dt}.$$

Similarly, the translational affine moment of forces has the following form:

$$N_{\mathrm{tr}}(\mathcal{O})^{ij} = x^i F^j.$$

The doubled skew-symmetric parts of the above quantities, i.e.,

$$S^{ij} = K^{ij} - K^{ji},$$

$$L_{\mathrm{tr}}(\mathcal{O})^{ij} = K_{\mathrm{tr}}(\mathcal{O})^{ij} - K_{\mathrm{tr}}(\mathcal{O})^{ji},$$
$$\mathcal{J}(\mathcal{O})^{ij} = K(\mathcal{O})^{ij} - K(\mathcal{O})^{ji},$$
$$\mathcal{N}^{ij} = N^{ij} - N^{ji},$$
$$\mathcal{N}_{\mathrm{tr}}(\mathcal{O})^{ij} = N_{\mathrm{tr}}(\mathcal{O})^{ij} - N_{\mathrm{tr}}(\mathcal{O})^{ji},$$
$$\mathcal{N}(\mathcal{O})^{ij} = N(\mathcal{O})^{ij} - N(\mathcal{O})^{ji},$$

represent the kinematical angular momentum and the moment of forces (torque). They also occur in three versions concerning, respectively, the internal motion (thus S is spin), motion of the centre of mass with respect to $\mathcal{O}$, and the total motion with respect to $\mathcal{O}$.

3.3 Affine constraints

If now we assume that the motion is affine, then the above expressions simplify to

$$K^{ij} = \varphi^i{}_A \frac{d\varphi^j{}_B}{dt} J^{AB},$$

$$K(\mathcal{O})^{ij} = K_{\mathrm{tr}}(\mathcal{O})^{ij} + K^{ij} = m x^i \frac{dx^j}{dt} + \varphi^i{}_A \frac{d\varphi^j{}_B}{dt} J^{AB},$$
$$N(\mathcal{O})^{ij} = N_{\mathrm{tr}}(\mathcal{O})^{ij} + N^{ij} = x^i F^j + N^{ij}.$$

Obviously, F and N become now functions of t, x^i, dx^i/dt, $\varphi^i{}_A$, $d\varphi^i{}_A/dt$. Let us stress that, just as it was the case with the kinetic energy, the above additive splitting into translational and internal parts is based on the assumption that the current centre of mass has permanently Lagrangian coordinates $a^K = 0$. This is consistent because barycenters are invariants of affine transformations.

By summation of elementary time rates of work over the body constituents, one can show that in the affine motion the total rate is given by

$$\mathcal{P} = g_{ij}\frac{dx^i}{dt}F^j + g_{ij}\Omega^i{}_k N^{kj}.$$

Let us remind however that, besides of active generalized forces F, N controlling affine modes of motion, there are also hidden structural forces keeping affine constraints, i.e., reactions. Their density $\mathcal{F}_R$ does not vanish, however, their monopole and dipole moments F_R, N_R do because, according to the d'Alembert principle, the reaction time rate of work vanishes for any constraints-compatible virtual velocities, i.e., for any possible dx^i/dt, Ω:

$$\mathcal{P}_R = g_{ij}\frac{dx^i}{dt}F_R{}^j + g_{ij}\Omega^i{}_k N_R{}^{kj} = 0.$$

Therefore, the effective reaction-free equations of motion are obtained from the primary non-constrained system by calculating the monopole and dipole moments.

The above derivation is quite general and valid for all kinds of forces, including non-potential and dissipative ones. It relies only on the metric structure g in M and on the d'Alembert principle. Obviously, if equations of motion follow from the Lagrangian $L = T - V(x, e)$, T given by (2.1), then the above analysis implies equations (3.4).

Similarly, one can easily show that

$$K^{ij} = \Sigma^i{}_m g^{mj}, \qquad k^i = g^{ij}p_j. \tag{3.5}$$

But these relationships become false when Lagrangian depends on velocities not only through the kinetic energy T but also through some generalized potential V, e.g., when magnetic or gyroscopic external forces are present. This is one of reasons we avoid denoting K^{ij} by Σ^{ij} or $K^i{}_j$ by $\Sigma^i{}_j$.

3.4 Balance equations

Kinematical quantities k^i, K^{ij} are intuitive because of their direct operational interpretation in terms of positions and velocities. At the same time they are lowest-order multipoles (monopoles and dipoles) of the distribution of kinematical linear momentum within the body, and it is difficult to over-estimate the application of multipoles and all moment quantities in practical problems of mechanics and field theory (cf, e.g., all Galerkin-type procedures [80]). On the other hand, their canonical counterparts p_i, $\Sigma^i{}_j$ have a very deep geometrical interpretation as Hamiltonian generators of fundamental transformation groups. Because of this, they are very often important constants of motions. In mechanics of affinely-rigid body, equations of motion are equivalent to the balance laws for $p_i, \Sigma^i{}_j$ or, in a sense equivalently, to the ones for k^i, K^{ij}, because Lagrangians of non-dissipative models, or at least Lagrangians of non-dissipative background dynamics, establish some link between these concepts. Similarly, in rigid-body mechanics equations of motion are equivalent to the balance for $p_i, (\Sigma^i{}_j - g^{ik}g_{jm}\Sigma^m{}_k)$ or for k^i, S^{ij}.

Equations of motion (3.3) may be written in several mutually equivalent balance forms. Let us quote some of them based on kinematical quantities like k^i, K^{ij} or their co-moving representation $\hat{k}^A, \hat{K}^{AB}$, where, obviously,

$$k^i = \varphi^i{}_A \hat{k}^A, \qquad K^{ij} = \varphi^i{}_A \varphi^j{}_B \hat{K}^{AB}.$$

The co-moving components $\hat{F}^A, \hat{N}^{AB}$ of generalized forces are given by analogous expressions, thus:

$$F^i = \varphi^i{}_A \hat{F}^A, \qquad N^{ij} = \varphi^i{}_A \varphi^j{}_B \hat{N}^{AB}.$$

The dynamical balance expressed in terms of kinematical (non-canonical) quantities in spatial (Eulerian) representation reads:

$$\frac{dk^i}{dt} = F^i, \qquad \frac{dK^{ij}}{dt} = \frac{d\varphi^i{}_A}{dt}\frac{d\varphi^j{}_B}{dt} J^{AB} + N^{ij}. \tag{3.6}$$

For non-dissipative potential systems with Lagrangians $L = T - V(x, \varphi)$, it reduces to (3.1), because then (3.4) holds. Let us observe that even in the interaction-free case, when $N = 0$, the balance for K is not a conservation law due to the first non-dynamical term on its right-hand side. One can

write that

$$\frac{dK^{ij}}{dt} = N^{ij} + 2\frac{\partial T_{\rm int}}{\partial g_{ij}}.$$

On the Hamiltonian level, this means that the non-conservation of K even in geodetic motion is due to the fact that the kinetic energy depends explicitly on the spatial metric tensor. Affine symmetry of degrees of freedom is broken and reduced to the Euclidean one.

The system (3.6) may be written in the following form:

$$\frac{dk^i}{dt} = F^i, \qquad \frac{dK(\mathcal{O})^{ij}}{dt} = m\frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{d\varphi^i{}_A}{dt}\frac{d\varphi^j{}_B}{dt}J^{AB} + N(\mathcal{O})^{ij}, \quad (3.7)$$

as a balance for the kinematical linear momentum and the total affine momentum with respect to some space-fixed origin $\mathcal{O} \in M$.

If the body is rigid in the usual metrical sense, i.e., all distances between its constituents are constant, then the d'Alembert principle implies that the second subsystems in (3.6), (3.7) are to be replaced by their skew-symmetric parts, thus,

$$\frac{dS^{ij}}{dt} = \mathcal{N}^{ij}, \qquad \frac{d\mathcal{J}(\mathcal{O})^{ij}}{dt} = \mathcal{N}(\mathcal{O})^{ij}.$$

To these equations the rigidity condition, i.e.,

$$\eta_{AB} = g_{ij}\varphi^i{}_A\varphi^j{}_B,$$

may be automatically substituted without paying any attention to reaction forces responsible for the metrical rigidity.

The above balance laws for kinematical angular momenta become conservation laws in the interaction-free case, and even under weaker, realistic conditions that N or $N(\mathcal{O})$ is symmetric. This is very natural on the Hamiltonian level, because S, $\mathcal{J}(\mathcal{O})$ are then directly related to their canonical counterparts. The latter are Hamiltonian generators of the isometry group of (M, g), thus, according to the Noether theorem, they are constants of motion in geodetic problems, because gyroscopic kinetic energy is isometry-invariant.

3.5 Affine counterparts of Euler equations

Another very convenient balance form of equations of motion is obtained when generalized velocities $d\varphi^i{}_A/dt$ are expressed through the non-holono-

mic quantities $\Omega^i{}_j$, then

$$\frac{dk^i}{dt} = F^i, \qquad \frac{dK^{ij}}{dt} = \Omega^i{}_m K^{mj} + N^{ij}. \tag{3.8}$$

Expressing our balance in co-moving (material) terms we obtain

$$\begin{aligned} \frac{d\hat{k}^A}{dt} &= -\hat{k}^B \widetilde{J}_{BC} \hat{K}^{CA} + \hat{F}^A, \\ \frac{d\hat{K}^{AB}}{dt} &= -\hat{K}^{AC} \widetilde{J}_{CD} \hat{K}^{DB} + \hat{N}^{AB}, \end{aligned} \tag{3.9}$$

or, using non-holonomic velocities,

$$\begin{aligned} m\frac{d\hat{v}^A}{dt} &= -m\hat{\Omega}^A{}_B \hat{v}^B + \hat{F}^A, \\ J^{AC}\frac{d\hat{\Omega}^B{}_C}{dt} &= -\hat{\Omega}^B{}_D \hat{\Omega}^D{}_C J^{CA} + \hat{N}^{AB}. \end{aligned}$$

It is a nice feature of the co-moving representation that all non-dynamical terms are built only of expressions $\hat{k}^A, \hat{K}^{AB}$ or $\hat{v}^A, \hat{\Omega}^A{}_B$ without any direct using of mixed quantities like $\varphi^i_A, d\varphi^i{}_A/dt$. The second (internal) subsystems are exactly affine counterparts of gyroscopic Euler equations and exactly reduce to them when the rigid-body constraints are imposed. The relationship between two co-moving forms is based on the equation

$$\hat{K}^{AB} = \hat{\Omega}^B{}_C J^{CA}$$

following directly from the definition of K. There is some relationship between this formula and Legendre transformation for Lagrangians $L = T - V(x, \varphi)$. Namely, one can show that the internal part of (3.2) may be equivalently written in the following form:

$$\hat{\Sigma}^A{}_B = \hat{K}^{AC} G_{CB} = G_{BC} \hat{\Omega}^C{}_D J^{DA}, \tag{3.10}$$

where $G \in U^* \otimes U^*$ denotes as previously the Green deformation tensor. Therefore, the canonical affine spin is obtained from the kinematical one by the G-lowering of the second index. As we saw, there was a similar formula (3.5) in the spatial representation, i.e.,

$$\Sigma^i{}_j = K^{im} g_{mj}. \tag{3.11}$$

It is seen that there is an easy possibility of confusion. Namely a superficial analogy with the last formula might suggest us to use the η-shifting of indices for establishing the Legendre link between K and Σ. However, for any reasonable Lagrangian

$$\hat{\Sigma}^A{}_B \neq \hat{K}^{AC}\eta_{CB}$$

except the special case of metrically-rigid motion. This is an additional reason for avoiding ambiguous symbols like $\hat{\Sigma}^{AB}$ or $\hat{K}^A{}_B$. More generally, if some tensor objects in V are related to each other by the g-shifting of indices, then the corresponding co-moving objects in U are interrelated by the G-shifting. And conversely, if two tensors in U are interrelated by the η-shift of indices, then their spatial counterparts in V are obtained from each other by the C-shifting, where $C \in V^* \otimes V^*$ is the Cauchy deformation tensor. The contravariant inverses of G and C are carefully denoted by $\widetilde{G} \in U \otimes U$, and $\widetilde{C} \in V \otimes V$,

$$\widetilde{G}^{AC}G_{CB} = \delta^A{}_B, \qquad \widetilde{C}^{ik}C_{kj} = \delta^i{}_j.$$

The notation G^{AB}, C^{ij} would be misleading because of the possible confusion with the objects

$$\eta^{AC}\eta^{BD}G_{CD}, \qquad g^{ik}g^{jm}C_{km}$$

obtained from G, C by the usual η- or g-metrical operations on indices.

For Lagrangian systems $L = T - V(x, \varphi)$ generalized forces Q, N are interrelated by (3.4), thus, just as in (3.11),

$$Q^i{}_j = N^{im}g_{mj}.$$

But in the co-moving representation, in analogy to (3.10), we have

$$\hat{Q}^A{}_B = \hat{N}^{AC}G_{CB}, \qquad \hat{Q}^A{}_B \neq \hat{N}^{AC}\eta_{CB}.$$

This has to do with different φ-transformation properties of Q, N:

$$Q^i{}_j = \varphi^i{}_A\hat{Q}^A{}_B(\varphi^{-1})^B{}_j, \qquad N^{ij} = \varphi^i{}_A\varphi^j{}_B\hat{N}^{AB};$$

similarly for $\Sigma^i{}_j$, K^{ij}, $\hat{\Sigma}^A{}_B$, $\hat{K}^{AB}$.

As seen from equations (3.8), (3.9) even in the interaction-free case neither K^{ij} nor $\hat{K}^{AB}$ are constants of motion. The same concerns their canonical counterparts $\Sigma^i{}_j$, $\hat{\Sigma}^A{}_B$. The reason is that the kinetic energy is not

invariant under spatial and material affine transformations (except translations, of course). At the same time, purely geodetic Hamiltonian models with $L = T$ are physically useless because, except of rest-states, all their trajectories (straight lines in $M\times$ LI(U, V)) escape to infinity. In particular, the body may expand to infinity and contract in finite time to a point. The metric on $Q = M\times$ LI(U, V) underlying the kinetic energy (2.1) is unable to encode realistic interactions and predict elastic vibrations in purely geodetic terms.

Chapter 4

Dynamical affine invariance

4.1 Dynamical affine models

Basing on the motivation presented in previous sections, we shall now consider some models which are ruled by affine groups not only on the kinematical but also on the dynamical level. In particular, we shall discuss left- and right-invariant Riemann metrics on linear and affine groups or rather, more precisely, on their free-action homogeneous spaces. We concentrate on geodetic models, when there is no potential term and the structure of interactions is encoded in an appropriately chosen metric tensor on the configuration space.

Let us begin with the internal sector, when translational degrees of freedom are frozen and the configuration space reduces to $Q_{\mathrm{int}} = M \times \mathrm{LI}(U, V)$, or equivalently to $\mathrm{F}(V)$ (when for simplicity we put $U = \mathbb{R}^n$). According to the transformation rules for Ω, $\hat{\Omega}$, the most general metric tensor on Q_{int} invariant under the action of $\mathrm{GL}(V)$ through (2.4) is that underlying the kinetic energy form given by

$$T_{\mathrm{int}} = \frac{1}{2}\mathcal{L}^{B}{}_{A}{}^{D}{}_{C}\hat{\Omega}^{A}{}_{B}\hat{\Omega}^{C}{}_{D}, \tag{4.1}$$

where coefficients $\mathcal{L}$ are constant and symmetric in bi-indices $({}^{B}{}_{A})$, $({}^{D}{}_{C})$.

This quadratic form is also assumed to be non-degenerate, although not necessarily positively definite. As $\hat{\Omega}$ is a non-holonomic velocity, i.e., it is not a time derivative of any system of generalized coordinates, the underlying metric on $Q_{\rm int}$ is curved.

Quite similarly, the most general kinetic energy invariant under material affine transformations (2.5) has the following form:

$$T_{\rm int} = \frac{1}{2}\mathcal{R}^{j}{}_{i}{}^{l}{}_{k}\Omega^{i}{}_{j}\Omega^{k}{}_{l}, \tag{4.2}$$

where $\mathcal{R}$ is also constant and symmetric in bi-indices $({}^{j}{}_{i})$, $({}^{l}{}_{k})$. The underlying metric tensor on $Q_{\rm int}$ is also curved, i.e., essentially Riemannian.

In general, (4.1) is not right, i.e., materially, invariant under $\mathrm{GL}(U)$ acting through (2.5), and (4.2) is not invariant under $\mathrm{GL}(V)$ acting through (2.4) in the physical space, i.e., on the left. The exceptional situation of simultaneous spatial and material invariance leads us to

$$T_{\rm int} = \frac{A}{2}\Omega^{i}{}_{j}\Omega^{j}{}_{i} + \frac{B}{2}\Omega^{i}{}_{i}\Omega^{j}{}_{j} = \frac{A}{2}\hat{\Omega}^{K}{}_{L}\hat{\Omega}^{L}{}_{K} + \frac{B}{2}\hat{\Omega}^{K}{}_{K}\hat{\Omega}^{L}{}_{L},$$

where A, B are some constants. Using invariant terms, we can say that such $T_{\rm int}$ is a linear combination of two basic second-order Casimir invariants, i.e.,

$$T_{\rm int} = \frac{A}{2}\mathrm{Tr}(\Omega^2) + \frac{B}{2}(\mathrm{Tr}\Omega)^2 = \frac{A}{2}\mathrm{Tr}(\hat{\Omega}^2) + \frac{B}{2}(\mathrm{Tr}\hat{\Omega})^2. \tag{4.3}$$

Such $T_{\rm int}$ is never positively-definite. The reason is that the maximal semisimple subgroups $\mathrm{SL}(V)$, $\mathrm{SL}(U)$ (their determinants equal to unity) are non-compact, thus, the quadratic form $\mathrm{Tr}(\Omega^2) = \mathrm{Tr}(\hat{\Omega}^2)$ has the hyperbolic signature $(n(n+1)/2\,+,\, n(n-1)/2\,-)$, where the positive contribution corresponds to the "non-compact" dimensions and the negative one to the "compact" dimensions in $\mathrm{GL}(V)$, $\mathrm{GL}(U)$.

By the way, the above quadratic forms reduce to the Killing form (Killing scalar products) on $\mathrm{L}(V)$, $\mathrm{L}(U)$ [32, 33, 34] when $A = 2n$, $B = -2$. As $\mathrm{L}(V)$, $\mathrm{L}(U)$ are non-semisimple, in this special unhappy case the scalar product (kinetic energy) is degenerate, thus, non-applicable in usual mechanical problems. The singularity consists of dilatational Lie algebras $\mathbb{R}\mathrm{Id}_V$, $\mathbb{R}\mathrm{Id}_U$. More generally, the same holds when $A = -Bn$. Paradoxically enough, non-degenerate forms (4.3) ($A \neq -Bn$) may be mechanically useful in spite of their non-definiteness.

4.2 The d'Alembert-like models

The usual d'Alembert model (2.1) invariant under additive translations (2.3) is the special case of general models of the following form:

$$T_{\rm int} = \frac{1}{2}\mathcal{A}^{K}{}_{i}{}^{L}{}_{j}\frac{d\varphi^{i}{}_{K}}{dt}\frac{d\varphi^{j}{}_{L}}{dt}, \tag{4.4}$$

where $\mathcal{A}$ is constant and symmetric in bi-indices $({}^{K}{}_{i})$, $({}^{L}{}_{j})$. The peculiarity of (2.1) within this class is that $\mathcal{A}$ factorizes, i.e.,

$$\mathcal{A}^{K}{}_{i}{}^{L}{}_{j} = g_{ij}J^{KL}, \tag{4.5}$$

and is invariant under the left action of $\mathrm{SO}(V,g)$ and the right action of $\mathrm{SO}(U,\widetilde{J})$ (in particular, $\mathrm{SO}(U,\eta)$, when the inertia is isotropic, i.e., $J=\mu\eta$). It is clear that the $\mathcal{A}$-based models of $T_{\rm int}$ are never invariant under $\mathrm{GL}(V)$, $\mathrm{GL}(U)$. The underlying metric on $\mathrm{LI}(U,V)$ is flat.

4.3 Translational motion

Let us now consider the translational sector of motion. The only model of translational kinetic energy invariant under $\mathrm{GAf}(M)$ (affine group of M) has the following form:

$$T_{\rm tr} = \frac{m}{2}C_{ij}\frac{dx^{i}}{dt}\frac{dx^{j}}{dt} = \frac{m}{2}\eta_{AB}\hat{v}^{A}\hat{v}^{B}. \tag{4.6}$$

It looks like the usual kinetic energy, however, there is a very essential difference. Namely, in the above expression the velocity vector is not squared with the help of the constant and absolutely fixed spacial metric $g\in V^{*}\otimes V^{*}$. Instead of it, the Cauchy deformation tensor C is used as an instantaneous metric tensor of M. Being a function of the internal configuration $\varphi\in\mathrm{LI}(U,V)$, it depends on time. It is so as if the instantaneous internal configuration created a dynamical metric in an essentially amorphous affine space M. In this sense the model is an oversimplified toy simulation of general relativity. At the same time it is clear that $T_{\rm tr}$ is not invariant under $\mathrm{GL}(U)$ because the fixed material metric η restricts the symmetry to $\mathrm{O}(U,\eta)$ (η-rotations of U).

If we wish the translational kinetic energy to be $\mathrm{GL}(U)$-invariant, then the only reasonable model is the usual one, based on the fixed M-metric g, i.e.,

$$T_{\mathrm{tr}} = \frac{m}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = \frac{m}{2} G_{AB} \hat{v}^A \hat{v}^B. \tag{4.7}$$

It is impossible to construct any model of T_{tr} and $T = T_{\mathrm{tr}} + T_{\mathrm{int}}$, which would be purely affine both in M and N; in one of these spaces some metric structure must be assumed. Therefore, although T_{int} alone may be affine simultaneously in M and N, there are no reasons to stick to such models, the more so they are never positively-definite. These problems have to do with the non-existence of a doubly (left- and right-) invariant pseudo-Riemannian structure on the affine group $\mathrm{GAf}(n, \mathbb{R}) \simeq \mathrm{GL}(n, \mathbb{R}) \times_s \mathbb{R}^n$. Any doubly-invariant twice covariant tensor field on this group is degenerate. Therefore, it is reasonable to concentrate on kinetic energies which are affine in M and η-metrical in N or, conversely, g-metrical in M and affine in N. The corresponding geodetic models in Q have a maximal possible symmetry, being at the same time really true geodetic problems (non-singular underlying metric). Such models are special cases of (4.1), (4.2), (4.6), (4.7), thus, we start with some statements concerning the general case.

4.4 Legendre transformation

For the model (4.1), (4.6) affine in space and matrical in the material, Legendre transformation has the form:

$$\hat{\Sigma}^A{}_B = \mathcal{L}^A{}_B{}^C{}_D \hat{\Omega}^D{}_C, \quad \hat{p}_A = m \eta_{AB} \hat{v}^B, \tag{4.8}$$

where, obviously, the second equation may be rewritten as

$$p_i = m C_{ij} v^j. \tag{4.9}$$

The corresponding geodetic Hamiltonian is given by

$$\mathcal{T} = \mathcal{T}_{\mathrm{tr}} + \mathcal{T}_{\mathrm{int}},$$

where

$$\begin{aligned} \mathcal{T}_{\mathrm{tr}} &= \frac{1}{2m} \eta^{AB} \hat{p}_A \hat{p}_B = \frac{1}{2m} \widetilde{C}^{ij} p_i p_j, \\ \mathcal{T}_{\mathrm{int}} &= \frac{1}{2} \widetilde{\mathcal{L}}^A{}_B{}^C{}_D \hat{\Sigma}^B{}_A \hat{\Sigma}^D{}_C, \end{aligned}$$

and the symmetric bimatrix $\widetilde{\mathcal{L}}$ is reciprocal to $\mathcal{L}$,

$$\widetilde{\mathcal{L}}^{A}{}_{B}{}^{K}{}_{L}\mathcal{L}^{L}{}_{K}{}^{C}{}_{D} = \delta^{A}{}_{D}\delta^{C}{}_{B}.$$

For the model (4.2), (4.7), metrical in space and affine in the material, Legendre transformation may be represented as follows:

$$\Sigma^{i}{}_{j} = \mathcal{R}^{i}{}_{j}{}^{k}{}_{l}\Omega^{l}{}_{k}, \qquad p_i = mg_{ij}v^{j}, \tag{4.10}$$

where, in analogy to the $\mathcal{L}$-case, the second subsystem may be rewritten as

$$\hat{p}_A = mG_{AB}\hat{v}^{B}.$$

Inverting the Legendre transformation, we obtain geodetic Hamiltonian

$$\mathcal{T} = \mathcal{T}_{\rm tr} + \mathcal{T}_{\rm int},$$

where, dually to the $\mathcal{L}$-formulas,

$$\mathcal{T}_{\rm tr} = \frac{1}{2m}g^{ij}p_ip_j = \frac{1}{2m}\widetilde{G}^{AB}\hat{p}_A\hat{p}_B, \qquad \mathcal{T}_{\rm int} = \frac{1}{2}\widetilde{\mathcal{R}}^{i}{}_{j}{}^{k}{}_{l}\Sigma^{j}{}_{i}\Sigma^{l}{}_{k}.$$

Obviously again $\widetilde{\mathcal{R}}$ denotes the inverse bimatrix of $\mathcal{R}$,

$$\widetilde{\mathcal{R}}^{a}{}_{b}{}^{k}{}_{l}\mathcal{R}^{l}{}_{k}{}^{j}{}_{i} = \delta^{a}{}_{i}\delta^{j}{}_{b}.$$

4.5 Non-geodetic $\mathcal{L}$-models

Let us admit non-geodetic models of the form

$$L = T - V(x,\varphi), \qquad H = \mathcal{T} + V(x,\varphi),$$

where $V(x,\varphi)$ is a usual potential energy depending only on the indicated configuration variables. Then the balance equations for $\mathcal{L}$-models (affine in space, metrical in the material) read:

$$\frac{dp_i}{dt} = Q_i, \qquad \frac{d\Sigma^{i}{}_{j}}{dt} = -\frac{1}{m}\widetilde{C}^{ik}p_kp_j + Q^{i}{}_{j}, \tag{4.11}$$

with the same meaning of generalized forces as previously, i.e.,

$$Q_i = -\frac{\partial V}{\partial x^i}, \qquad Q^{i}{}_{j} = -\varphi^{i}{}_{A}\frac{\partial V}{\partial \varphi^{j}{}_{A}}.$$

When taken together with the Legendre transformation, the above balance laws are equivalent to the Hamilton canonical equations. They may be also generalized so as to include some non-Hamiltonian, e.g., dissipative terms on their right-hand sides. It is seen that in the purely geodetic case (when $Q_i = 0$, $Q^i{}_j = 0$) the canonical linear momentum is conserved, but the affine spin is not so due to the first term on the right-hand side of the balance law for Σ. The reason is that $\Sigma^i{}_j$ generate linear transformations of internal degrees of freedom; these transformations do not affect translational variables. Therefore, the full affine symmetry is broken, and Σ is not a constant of motion. But one can reformulate the balance laws (4.11), just as in the d'Alembert model, by introducing the total canonical affine momentum $I(\mathcal{O})$, related to some fixed origin $\mathcal{O} \in M$,

$$I(\mathcal{O})^i{}_j := \Lambda(\mathcal{O})^i{}_j + \Sigma^i{}_j = x^i p_j + \Sigma^i{}_j.$$

The first term refers to the translational motion, the second one to the relative (internal) motion. Something similar may be done for generalized forces,

$$Q_{\rm tot}(\mathcal{O})^i{}_j := Q_{\rm tr}(\mathcal{O})^i{}_j + Q^i{}_j = x^i Q_j + Q^i{}_j.$$

Then the system of balance equations (4.11) may be written in the following equivalent form:

$$\frac{dp_i}{dt} = Q_i, \qquad \frac{dI(\mathcal{O})^i{}_j}{dt} = Q_{\rm tot}(\mathcal{O})^i{}_j. \tag{4.12}$$

It is seen that in the geodetic case one obtains conservation laws for p_i, $I(\mathcal{O})^i{}_j$, i.e., for the system of generators of the spatial affine group $\mathrm{GAf}(M)$, just as expected.

4.6 "Drunk missile" effect

Let us observe some funny feature of our geodetic equations. Namely, the canonical linear momentum is a constant of motion, but the translational velocity is not. This is because the Legendre transformation (4.9) implies that the translational motion is influenced by internal phenomena. Except some special solutions even the direction of translational velocity is non-constant and depends on what happens with "internal" degrees of freedom. This is a kind of "drunk missile" effect. Something similar occurs in the dynamics

of defects in solids [35]. It is also non-excluded that the non-conservation of velocity might be an over-simplified model of certain specially-relativistic phenomena (internal motion results in changes of internal energy, and therefore, in the rest mass pulsations; but the latter ones influence the effective inertia, and therefore, the translational motion). One can show that the time-rate of translational velocity may be expressed as follows:

$$m\frac{dv^a}{dt} = -\widetilde{C}^{aj}\frac{dC_{jb}}{dt}v^b + F^a = mv^b(\Omega^a{}_b + \widetilde{C}^{ad}\Omega^m{}_d C_{mb}) + F^a,$$

where the contravariant force F is given by the following expression:

$$F^a = \widetilde{C}^{ab}Q_b$$

(because of the formerly mentioned reasons, we avoid denoting F^a as Q^a). It is explicitly seen that v is variable even in the purely geodetic motion. The $\mathcal{L}$-based geodetics in $M\times$ LI(U,V) do not project onto straight lines in M.

4.7 Non-geodetic $\mathcal{R}$-models

Let us now consider the $\mathcal{R}$-based models (4.2), (4.7), metrical in space and affine in the material. As mentioned, they are somehow related to the Arnold-Ebin-Marsden-Binz approach to the dynamics of ideal fluids [1, 2, 7, 17, 18, 42, 43]. Our Poisson brackets imply that the balance form of equations of motion may be expressed as follows:

$$\frac{dp_a}{dt} = Q_a, \qquad \frac{d\hat{\Sigma}^A{}_B}{dt} = \hat{Q}^A{}_B, \tag{4.13}$$

again with

$$Q_a = -\frac{\partial V}{\partial x^a}, \qquad \hat{Q}^A{}_B = -\frac{\partial V}{\partial \varphi^i{}_A}\varphi^i{}_B = (\varphi^{-1})^A{}_i Q^i{}_j \varphi^j{}_B$$

in the potential case. In geodetic models p_a, $\hat{\Sigma}^A{}_B$ are conserved quantities as explicitly seen from the balance equations and expected on the basis of invariance properties. Indeed, the $\mathcal{R}$-model of T is invariant under the Abelian group of translations in M. Therefore, p_a are constants of motion as Hamiltonian generators of this group. Similarly, as seen from our

Poisson brackets, $\hat{\Sigma}^A{}_B$ Poisson-commute with p_a and $\Sigma^i{}_j$, therefore, with the total geodetic Hamiltonian $\mathcal{T}$. This is due to the explicitly obvious invariance of the $\mathcal{R}$-based T under the group of material linear transformations $\mathrm{GL}(U)$. Surprisingly enough, the co-moving components of linear momentum, $\hat{p}_A = p_i\varphi^i{}_A$, are not constants of motion. But this is also clear because the material space N has in our model a distinguished point, i.e., the Lagrangian position of the centre of mass. Because of this, translations in N fail to be symmetries and their Hamiltonian generators $\hat{p}_A$ are non-conserved. According to the structure of Legendre transformation, translational velocity

$$v^a = \frac{dx^a}{dt} = g^{ab}p_b$$

is also a constant of motion, just as p itself. Therefore, in geodetic $\mathcal{R}$-models, the geodesic curves in $M\times \mathrm{LI}(U,V)$ project to M onto straight lines swept with constant velocities (uniform motions). This means that there is no "drunk missile effect" and contravariant representation of the translational balance takes on the usual form:

$$m\frac{dv^a}{dt} = m\frac{d^2x^a}{dt^2} = F^a, \qquad F^a = g^{ab}Q_b.$$

4.8 Structure of equations of motion

As previously, the balance laws (4.13) become a closed system of equations of motion when considered jointly with the Legendre transformation (4.10). Let us observe that the structure of (4.13) is in a sense less "aesthetical" than that of (4.11) because it is more non-homogeneous. The point is that in (4.11) both subsystems are written in terms of spatial objects, whereas in (4.13) one uses the mixed representation: spatial for the translational motion and material for the internal one. Obviously, (4.13) may be done symmetric, dual to (4.11), by substituting $\hat{p}_A = p_i\varphi^i{}_A$. But this immediately makes the translational equation more complicated.

There is also another problem. The simplicity of our balance laws (conservation laws in the geodetic case) is rather illusory. The point is that, as mentioned above, the total system of equations of motion consists of the balance laws and Legendre transformation. The balance (4.11), (4.12) looks simple in Euler (spatial) representation, but the corresponding Legendre transformation is simple in Lagrangian (material) representation (4.8).

And quite conversely, the internal part of (4.13) is simple in the co-moving terms, but its Legendre transformation is simple when expressed in the spatial (Eulerian) form (4.10).

One can easily show that the internal parts of Legendre transformations (4.8), (4.10) may be respectively expressed as follows:

$$\Sigma^i{}_j = \check{\mathcal{L}}^i{}_j{}^k{}_l \Omega^l{}_k, \tag{4.14}$$

$$\hat{\Sigma}^A{}_B = \hat{\mathcal{R}}^A{}_B{}^C{}_D \hat{\Omega}^D{}_C, \tag{4.15}$$

where

$$\check{\mathcal{L}}^i{}_j{}^k{}_l = \varphi^i{}_A (\varphi^{-1})^B{}_j \varphi^k{}_C (\varphi^{-1})^D{}_l \mathcal{L}^A{}_B{}^C{}_D,$$

$$\hat{\mathcal{R}}^A{}_B{}^C{}_D = (\varphi^{-1})^A{}_i \varphi^j{}_B (\varphi^{-1})^C{}_k \varphi^l{}_D \mathcal{R}^i{}_j{}^k{}_l.$$

Obviously, this form is rather complicated because the coefficients at Ω and $\hat{\Omega}$ are non-constant; they depend on the internal configuration φ. Simplicity of the balance laws is incompatible with simplicity of Legendre transformations.

4.9 Internal motion

As mentioned, when translational degrees of freedom are taken into account, there are no sensible models which would be affine simultaneously in space and in the material. The highest symmetry of mathematical interest and at the same time physically reasonable is that affine in space and rotational in the material, and conversely, Euclidean in space and (centro-)affine in the body. The latter model is an over-simplified discretization of dynamical systems on diffeomorphisms group as used in hydrodynamics and elasticity.

In materially isotropic $\mathcal{L}$-models the quantity $\mathcal{L}^A{}_B{}^C{}_D$ is a linear combination of tensors $\eta^{AC}\eta_{BD}$, $\delta^A{}_D\delta^C{}_B$, $\delta^A{}_B\delta^C{}_D$. Similarly, in spatially isotropic $\mathcal{R}$-models the tensor $\mathcal{R}^i{}_j{}^k{}_l$ is a linear combination of terms $g^{ik}g_{jl}$, $\delta^i{}_l\delta^k{}_j$, $\delta^i{}_j\delta^k{}_l$. Therefore, (4.1), (4.2) take on, respectively, the following forms:

$$T_{\text{int}} = \frac{I}{2}\eta_{KL}\hat{\Omega}^K{}_M\hat{\Omega}^L{}_N\eta^{MN} + \frac{A}{2}\hat{\Omega}^K{}_L\hat{\Omega}^L{}_K + \frac{B}{2}\hat{\Omega}^K{}_K\hat{\Omega}^L{}_L, \tag{4.16}$$

$$T_{\text{int}} = \frac{I}{2}g_{ik}\Omega^i{}_j\Omega^k{}_l g^{jl} + \frac{A}{2}\Omega^i{}_j\Omega^j{}_i + \frac{B}{2}\Omega^i{}_i\Omega^j{}_j, \tag{4.17}$$

where the constants I, A, B are generalized internal inertia scalars. It is clear that if $I = 0$, then these expressions become identical. The I-terms break the centro-affine symmetry in U and V, and restrict it to the metrical one, respectively, in the sense of metric tensors η or g. The first term in (4.16), just as (4.6), may be expressed in terms of the Cauchy deformation tensor, i.e.,

$$\frac{I}{2}C_{ij}\frac{d\varphi^i{}_A}{dt}\frac{d\varphi^j{}_B}{dt}\eta^{AB}. \tag{4.18}$$

Let us observe that the isotropic inertial tensor $I\eta^{AB}$ in (4.18) might be replaced by the general one,

$$\frac{1}{2}C_{ij}\frac{d\varphi^i{}_A}{dt}\frac{d\varphi^j{}_B}{dt}J^{AB} = \frac{1}{2}\eta_{KL}\hat{\Omega}^K{}_A\hat{\Omega}^L{}_B J^{AB}. \tag{4.19}$$

This expression is structurally similar to the d'Alembert formula (2.1). The difference is that the fixed metric g is replaced by the φ-dependent Cauchy tensor C. There is not only formal similarity but also some asymptotic correspondence between (4.19) and (2.1). Obviously, for the general J, (4.19) is not metrically isotropic and its internal symmetry is reduced to $\mathrm{O}(U,\eta)\cap\mathrm{O}(U,\widetilde{J})$. The I-terms in (4.16), (4.17) are positively definite if $I > 0$. Moreover, the total expressions (4.16), (4.17) are positively definite for some open range of $(I, A, B) \in \mathbb{R}^3$. Roughly speaking, the absolute values of A, B must be "sufficiently small" in comparison with I.

4.10 $\mathcal{L}$-models

The internal part of Legendre transformation (4.8) for $\mathcal{L}$-models becomes now (i.e., for (4.16)):

$$\hat{\Sigma}^K{}_L = I\eta^{KM}\eta_{LN}\hat{\Omega}^N{}_M + A\hat{\Omega}^K{}_L + B\delta^K{}_L\hat{\Omega}^M{}_M. \tag{4.20}$$

This may be alternatively written as follows:

$$\Sigma^i{}_j = I\widetilde{C}^{ib}C_{ja}\Omega^a{}_b + A\Omega^i{}_j + B\delta^i{}_j\Omega^m{}_m,$$

it is the very special case of (4.14).

The inverse Legendre transformation has the same structure, i.e.,

$$\hat{\Omega}^K{}_L = \frac{1}{\widetilde{I}}\eta^{KM}\eta_{LN}\hat{\Sigma}^N{}_M + \frac{1}{\widetilde{A}}\hat{\Sigma}^K{}_L + \frac{1}{\widetilde{B}}\delta^K{}_L\hat{\Sigma}^M{}_M, \tag{4.21}$$

where

$$\widetilde{I} = \frac{\left(I^2 - A^2\right)}{I}, \qquad \widetilde{A} = \frac{\left(A^2 - I^2\right)}{A}, \qquad \widetilde{B} = -\frac{(I + A)\,(I + A + nB)}{B}.$$

When written in Eulerian (spatial) terms, this formula becomes

$$\Omega^i{}_j = \frac{1}{\widetilde{I}}\widetilde{C}^{ib}C_{ja}\Sigma^a{}_b + \frac{1}{\widetilde{A}}\Sigma^i{}_j + \frac{1}{\widetilde{B}}\delta^i{}_j\Sigma^m{}_m,$$

with the same as previously meaning of modified inertial coefficients $\widetilde{I}$, $\widetilde{A}$, $\widetilde{B}$.

4.11 $\mathcal{R}$-models

Similarly, for $\mathcal{R}$-models based on (4.17) the internal sector of Legendre transformation has the following form

$$\Sigma^i{}_j = Ig^{im}g_{jn}\Omega^n{}_m + A\Omega^i{}_j + B\delta^i{}_j\Omega^m{}_m, \tag{4.22}$$

which is inverted as

$$\Omega^i{}_j = \frac{1}{\widetilde{I}}g^{im}g_{jn}\Sigma^n{}_m + \frac{1}{\widetilde{A}}\Sigma^i{}_j + \frac{1}{\widetilde{B}}\delta^i{}_j\Sigma^m{}_m. \tag{4.23}$$

The co-moving representation of these formulas is given by the following expressions:

$$\begin{aligned}
\hat{\Sigma}^K{}_L &= I\widetilde{G}^{KM}G_{LN}\hat{\Omega}^N{}_M + A\hat{\Omega}^K{}_L + B\delta^K{}_L\hat{\Omega}^M{}_M,\\
\hat{\Omega}^K{}_L &= \frac{1}{\widetilde{I}}\widetilde{G}^{KM}G_{LN}\hat{\Sigma}^N{}_M + \frac{1}{\widetilde{A}}\hat{\Sigma}^K{}_L + \frac{1}{\widetilde{B}}\delta^K{}_L\hat{\Sigma}^M{}_M.
\end{aligned}$$

4.12 Balance laws

The general balance laws (4.11), (4.13) considered jointly with these Legendre transformations (including the obvious translational sector) provide the complete system of equations of motion (naturally, the definitions of Ω, $\hat{\Omega}$ are to be substituted). These equations, due to the very special structure of (4.16), (4.17) are relatively readible and effective. At the same time one can show that for incompressible bodies even in the purely geodetic case

($Q_i = 0$, $Q^i{}_j = 0$) there exists an open set of solutions which are bounded in the internal configuration space $\mathrm{LI}(U, V)$, so the elastic vibrations may be encoded in the very kinetic energy (Riemann structure) without the explicite use of forces.

4.13 Hamiltonians of internal motion

After substituting the above inverse of Legendre transformations to kinetic energies (4.16), (4.17), we obtain the following geodetic Hamiltonians of internal motion:

$$\mathcal{T}_{\mathrm{int}} = \frac{1}{2\widetilde{I}}\eta_{KL}\hat{\Sigma}^K{}_M\hat{\Sigma}^L{}_N\eta^{MN} + \frac{1}{2\widetilde{A}}\hat{\Sigma}^K{}_L\hat{\Sigma}^L{}_K + \frac{1}{2\widetilde{B}}\hat{\Sigma}^K{}_K\hat{\Sigma}^L{}_L, \tag{4.24}$$

$$\mathcal{T}_{\mathrm{int}} = \frac{1}{2\widetilde{I}}g_{ik}\Sigma^i{}_j\Sigma^k{}_l g^{jl} + \frac{1}{2\widetilde{A}}\Sigma^i{}_j\Sigma^j{}_i + \frac{1}{2\widetilde{B}}\Sigma^i{}_i\Sigma^j{}_j. \tag{4.25}$$

In certain problems it may be convenient to write down the first formula in spatial terms; similarly, the second one may be expressed with the use of co-moving representation. Therefore, we obtain, respectively, the following expressions:

$$\mathcal{T}_{\mathrm{int}} = \frac{1}{2\widetilde{I}}C_{kl}\Sigma^k{}_m\Sigma^l{}_n\widetilde{C}^{mn} + \frac{1}{2\widetilde{A}}\Sigma^k{}_l\Sigma^l{}_k + \frac{1}{2\widetilde{B}}\Sigma^k{}_k\Sigma^l{}_l,$$

$$\mathcal{T}_{\mathrm{int}} = \frac{1}{2\widetilde{I}}G_{KL}\hat{\Sigma}^K{}_M\hat{\Sigma}^L{}_N\widetilde{G}^{MN} + \frac{1}{2\widetilde{A}}\hat{\Sigma}^K{}_L\hat{\Sigma}^L{}_K + \frac{1}{2\widetilde{B}}\hat{\Sigma}^K{}_K\hat{\Sigma}^L{}_L.$$

The corresponding velocity-based formulas (4.16), (4.17) for kinetic energy may be written in an analogous way. Simply $1/\widetilde{I}$, $1/\widetilde{A}$, $1/\widetilde{B}$ in the last expressions are to be replaced by I, A, B, and simultaneously one must substitute $\Omega^k{}_l$, $\hat{\Omega}^K{}_L$ instead of $\Sigma^k{}_l$, $\hat{\Sigma}^K{}_L$.

For certain purposes it is convenient to rewrite geodetic Hamiltonians (4.24), (4.25) in an alternative form:

$$\mathcal{T}_{\mathrm{int}} = \frac{1}{2\alpha}\mathrm{Tr}(\hat{\Sigma}^2) + \frac{1}{2\beta}(\mathrm{Tr}\hat{\Sigma})^2 - \frac{1}{4\mu}\mathrm{Tr}(V^2), \tag{4.26}$$

$$\mathcal{T}_{\mathrm{int}} = \frac{1}{2\alpha}\mathrm{Tr}(\Sigma^2) + \frac{1}{2\beta}(\mathrm{Tr}\Sigma)^2 - \frac{1}{4\mu}\mathrm{Tr}(S^2), \tag{4.27}$$

where $\alpha := I + A$, $\beta := -(I + A)(I + A + nB)/B$, $\mu := (I^2 - A^2)/I$, and V, S denote, respectively, the vorticity and spin given by (2.6). It is clear that

the only distinction between expressions (4.26) and (4.27) is that in their third terms, thus, we can rewrite them concisely like

$$\mathcal{T}_{\rm int} = \frac{1}{2\alpha}C(2) + \frac{1}{2\beta}C(1)^2 - \frac{1}{4\mu}{\rm Tr}(V^2), \tag{4.28}$$

$$\mathcal{T}_{\rm int} = \frac{1}{2\alpha}C(2) + \frac{1}{2\beta}C(1)^2 - \frac{1}{4\mu}{\rm Tr}(S^2), \tag{4.29}$$

where $C(k)$ denotes the k-th order Casimir expression built of generators, i.e., $C(k) := {\rm Tr}(\Sigma^k) = {\rm Tr}(\hat{\Sigma}^k)$.

By analogy with the physical 3-dimensional case the quantity $-{\rm Tr}(S^2)$ may be interpreted as the doubled squared norm of the internal angular momentum, $-{\rm Tr}(S^2) = 2\|S\|^2$. Similarly, $-{\rm Tr}(V^2) = 2\|V\|^2$. Therefore, the formulas (4.28), (4.29) may be respectively written in the following intuitive and suggestive way:

$$\mathcal{T}_{\rm int} = \frac{1}{2\alpha}C(2) + \frac{1}{2\beta}C(1)^2 + \frac{1}{2\mu}\|V\|^2, \tag{4.30}$$

$$\mathcal{T}_{\rm int} = \frac{1}{2\alpha}C(2) + \frac{1}{2\beta}C(1)^2 + \frac{1}{2\mu}\|S\|^2. \tag{4.31}$$

Obviously, for the model (4.3) invariant under the left and right action of linear groups we have

$$\mathcal{T}_{\rm int} = \frac{1}{2A}C(2) + \frac{1}{2A(n + A/B)}C(1)^2. \tag{4.32}$$

4.14 Matrix representation

When performing computations, it is convenient to use orthogonal coordinates, $\eta_{KL} = \delta_{KL}$, $g_{ij} = \delta_{ij}$, and rewrite some of the above formulas in terms of purely matrix operations. Thus, (4.16), (4.17) become, respectively,

$$T_{\rm int} = \frac{I}{2}{\rm Tr}(\hat{\Omega}^T\hat{\Omega}) + \frac{A}{2}{\rm Tr}(\hat{\Omega}^2) + \frac{B}{2}({\rm Tr}\hat{\Omega})^2, \tag{4.33}$$

$$T_{\rm int} = \frac{I}{2}{\rm Tr}(\Omega^T\Omega) + \frac{A}{2}{\rm Tr}(\Omega^2) + \frac{B}{2}({\rm Tr}\Omega)^2. \tag{4.34}$$

Obviously, the second and third terms in these formulas are pairwise equal because ${\rm Tr}(\hat{\Omega}^2) = {\rm Tr}(\Omega^2)$ and ${\rm Tr}(\hat{\Omega}) = {\rm Tr}(\Omega)$.

Legendre transformations (4.20), (4.22) and their inverses (4.21), (4.23) are respectively given by the following expressions:

$$\hat{\Sigma} = I\hat{\Omega}^T + A\hat{\Omega} + B(\mathrm{Tr}\hat{\Omega})I_n, \qquad \Sigma = I\Omega^T + A\Omega + B(\mathrm{Tr}\Omega)I_n,$$

$$\hat{\Omega} = \frac{1}{\widetilde{I}}\hat{\Sigma}^T + \frac{1}{\widetilde{A}}\hat{\Sigma} + \frac{1}{\widetilde{B}}(\mathrm{Tr}\hat{\Sigma})I_n, \qquad \Omega = \frac{1}{\widetilde{I}}\Sigma^T + \frac{1}{\widetilde{A}}\Sigma + \frac{1}{\widetilde{B}}(\mathrm{Tr}\Sigma)I_n,$$

where I_n denotes the n-th order identity matrix.

Similarly, for the kinetic Hamiltonians (4.24), (4.25) we have, respectively,

$$\mathcal{T}_{\mathrm{int}} = \frac{1}{2\widetilde{I}}\mathrm{Tr}(\hat{\Sigma}^T\hat{\Sigma}) + \frac{1}{2\widetilde{A}}\mathrm{Tr}(\hat{\Sigma}^2) + \frac{1}{2\widetilde{B}}(\mathrm{Tr}\hat{\Sigma})^2, \tag{4.35}$$

$$\mathcal{T}_{\mathrm{int}} = \frac{1}{2\widetilde{I}}\mathrm{Tr}(\Sigma^T\Sigma) + \frac{1}{2\widetilde{A}}\mathrm{Tr}(\Sigma^2) + \frac{1}{2\widetilde{B}}(\mathrm{Tr}\Sigma)^2. \tag{4.36}$$

This matrix representation is very lucid and useful in calculations. Nevertheless, in comparison with the systematic tensor language, it may be risky and misleading because it obscures the geometric meaning of symbols and concepts. And this is worse than the lack of aesthetics; when no care is taken, this may lead simply to logical and numerical mistakes.

4.15 Geometrical remarks

We finish this section with some geometric remarks.

Kinetic energy T of a non-relativistic mechanical system is equivalent to some Riemannian structure Γ on its configuration space Q. In terms of generalized coordinates and velocities, we have that

$$T = \frac{1}{2}\Gamma_{\alpha\beta}(q)\frac{dq^\alpha}{dt}\frac{dq^\beta}{dt}, \qquad \Gamma = \Gamma_{\alpha\beta}(q)dq^\alpha \otimes dq^\beta.$$

Although usually somehow related to the metric tensor g of the physical space M, Γ need not be directly interpretable in terms of geometrical distances in M. As a rule, it depends not only on g but also on certain parameters characterizing inertial properties of the system, i.e., masses, inertial moments, etc.

It is instructive to describe explicitly in a bit more geometric form the metric tensors Γ on $Q = M \times \mathrm{LI}(U, V)$ underlying the kinetic energies defined

above. For this purpose it is convenient to introduce auxiliary geometric objects.

Let E_A, e_i denote the basic vectors in U, V underlying our Lagrange and Euler coordinates a^K, x^j. The corresponding dual basic covectors in U^*, V^* will be denoted as usual by E^A, e^i. Generalized coordinates in Q will be, as previously, denoted by x^i, $\varphi^i{}_A$, and no misunderstandings just simplifications follow from using the same symbol x^i for coordinates in M and for their pull-backs to Q. Now, we introduce two families of Pfaff forms (differential one-forms) on Q, i.e.,

$$\hat{\omega}^A{}_B := (\varphi^{-1})^A{}_i d\varphi^i{}_B, \qquad \omega^i{}_j := d\varphi^i{}_A(\varphi^{-1})^A{}_j.$$

These basic systems depend on the choice of bases E, e, but this is, so to speak, a covariant dependence. In other words the $\mathrm{L}(U)$- and $\mathrm{L}(V)$-valued one-forms

$$\hat{\omega} := \hat{\omega}^A{}_B E_A \otimes E^B, \qquad \omega := \omega^i{}_j e_i \otimes e^j$$

are base-independent.

In addition, we shall need the following two families of Pfaff forms:

$$\hat{\theta}^A = (\varphi^{-1})^A{}_i dx^i, \qquad \theta^i = \varphi^i{}_A \hat{\theta}^A = dx^i.$$

Just as previously, they give rise to the objective base-independent U- and V-valued differential one-forms:

$$\hat{\theta} = \hat{\theta}^A E_A, \qquad \theta = \theta^i e_i.$$

The base-independent objects $\hat{\omega}, \hat{\theta}$ and ω, θ could be defined without any use of bases, however, the above definitions are technically simplest. The above objects are closely related to the concept of affine velocity in co-moving (Lagrange) and spatial (Euler) representations. Namely, for any history $\mathbb{R} \ni t \mapsto (x(t), \varphi(t))$, the quantities $\hat{\Omega}^A{}_B$, $\Omega^i{}_j$ are evaluations of $\hat{\omega}^A{}_B$, $\omega^i{}_j$ on the tangent vectors (general velocities) given analytically by $dx^i/dt, d\varphi^i{}_A/dt$. Roughly speaking, we could say that

$$\hat{\Omega}^A{}_B = \frac{\hat{\omega}^A{}_B}{dt}, \qquad \Omega^i{}_j = \frac{\omega^i{}_j}{dt}.$$

This is obviously a kind of joke, but fully justified on the basis of infinitesimal Leibniz notation.

Similarly, the co-moving and spatial components of translational velocity are given by evaluations of $\hat{\theta}^A$ and θ^i on tangent vectors, and using the same trick we could say that

$$\hat{v}^A = \frac{\hat{\theta}^A}{dt}, \qquad v^i = \frac{\theta^i}{dt}.$$

At any point of the configuration space the systems $\theta^i, \omega^k{}_l$ and $\hat{\theta}^A, \hat{\omega}^K{}_L$ provide two bases in the corresponding space of covariant vectors. We could ask for the corresponding bases of contravariant vector spaces. It is convenient to use the language of contemporary differential geometry, where vector fields X with components X^i (meant in the sense of some local coordinates q^i) are identified with first-order differential operators of directional derivatives ∇_X, i.e.,

$$X = X^i \frac{\partial}{\partial q^i}.$$

One can easily show that the bases $\hat{H}_K, \hat{E}^A{}_B$ and $H_k, E^a{}_b$, dual respectively to $\hat{\theta}^K, \hat{\omega}^A{}_B$ and $\theta^k, \omega^a{}_b$, are given by the following differential operators:

$$\hat{H}_K = \varphi^i{}_K \frac{\partial}{\partial x^i}, \qquad \hat{E}^A{}_B = \varphi^k{}_B \frac{\partial}{\partial \varphi^k{}_A},$$

$$H_k = \frac{\partial}{\partial x^k}, \qquad E^a{}_b = \varphi^a{}_K \frac{\partial}{\partial \varphi^b{}_K}.$$

4.16 Affine models in geometric terms

The general $\mathcal{L}$-models (4.1), (4.6) are based on metric tensors of the following form:

$$\Gamma = m\eta_{AB}\hat{\theta}^A \otimes \hat{\theta}^B + \mathcal{L}^B{}_A{}^D{}_C \hat{\omega}^A{}_B \otimes \hat{\omega}^C{}_D.$$

Similarly, for $\mathcal{R}$-models (4.2), (4.7) we have

$$\Gamma = m g_{ij}\theta^i \otimes \theta^j + \mathcal{R}^j{}_i{}^l{}_k \omega^i{}_j \otimes \omega^k{}_l.$$

The corresponding contravariant metrics underlying the kinetic Hamiltonians are given, respectively, by

$$\widetilde{\Gamma} = \frac{1}{m}\eta^{AB}\hat{H}_A \otimes \hat{H}_B + \widetilde{\mathcal{L}}^B{}_A{}^D{}_C \hat{E}^A{}_B \otimes \hat{E}^C{}_D$$

for $\mathcal{L}$-models and

$$\widetilde{\Gamma} = \frac{1}{m} g^{ij} H_i \otimes H_j + \widetilde{\mathcal{R}}^{j}{}_{i}{}^{l}{}_{k} E^{i}{}_{j} \otimes E^{k}{}_{l}$$

for $\mathcal{R}$-models.

If the kinetic energy of internal motion (4.3) is invariant simultaneously under $\mathrm{GL}(V)$ and $\mathrm{GL}(U)$, then the corresponding metric tensor on $\mathrm{LI}(U,V)$ is given by

$$\Gamma^0_{\mathrm{int}} = A\hat{\omega}^{K}{}_{L} \otimes \hat{\omega}^{L}{}_{K} + B\hat{\omega}^{K}{}_{K} \otimes \hat{\omega}^{L}{}_{L} = A\omega^{k}{}_{l} \otimes \omega^{l}{}_{k} + B\omega^{k}{}_{k} \otimes \omega^{l}{}_{l}$$

and its inverse by

$$\begin{aligned} \widetilde{\Gamma}^0_{\mathrm{int}} &= \frac{1}{A} \hat{E}^{K}{}_{L} \otimes \hat{E}^{L}{}_{K} - \frac{B}{A(A+nB)} \hat{E}^{K}{}_{K} \otimes \hat{E}^{L}{}_{L} \\ &= \frac{1}{A} E^{k}{}_{l} \otimes E^{l}{}_{k} - \frac{B}{A(A+nB)} E^{k}{}_{k} \otimes E^{l}{}_{l}. \end{aligned}$$

Here Γ^0_{int} is a linear combination of two Casimir-like objects built of Pfaff forms ω in a quadratic way. As already mentioned, Γ^0_{int} becomes the group-theoretic Killing tensor when $A = 2n$, $B = -2$. This is just the pathological situation to be excluded because for $A/B = -n$ the tensor Γ^0_{int} is singular.

For our models affine in space and metrical in the body we have that

$$\Gamma = m\eta_{KL} \hat{\theta}^K \otimes \hat{\theta}^L + I\eta_{KL}\eta^{MN} \hat{\omega}^{K}{}_{M} \otimes \hat{\omega}^{L}{}_{N} + \Gamma^0_{\mathrm{int}}.$$

Similarly, for models metrical in space and affine in the body:

$$\Gamma = m g_{ij} \theta^i \otimes \theta^j + I g_{ik} g^{jl} \omega^{i}{}_{j} \otimes \omega^{k}{}_{l} + \Gamma^0_{\mathrm{int}}.$$

The corresponding contravariant (reciprocal) metrics are given by

$$\begin{aligned} \widetilde{\Gamma} &= \frac{1}{m} \eta^{KL} \hat{H}_K \otimes \hat{H}_L \\ &+ \frac{1}{\widetilde{I}} \eta_{KL}\eta^{MN} \hat{E}^{K}{}_{M} \otimes \hat{E}^{L}{}_{N} + \frac{1}{\widetilde{A}} \hat{E}^{K}{}_{L} \otimes \hat{E}^{L}{}_{K} + \frac{1}{\widetilde{B}} \hat{E}^{K}{}_{K} \otimes \hat{E}^{L}{}_{L} \end{aligned}$$

for spatially affine models and

$$\widetilde{\Gamma} = \frac{1}{m} g^{ij} H_i \otimes H_j + \frac{1}{\widetilde{I}} g_{ik} g^{jl} E^{i}{}_{j} \otimes E^{k}{}_{l} + \frac{1}{\widetilde{A}} E^{i}{}_{j} \otimes E^{j}{}_{i} + \frac{1}{\widetilde{B}} E^{k}{}_{k} \otimes E^{l}{}_{l}$$

for materially affine models. Obviously, the last two terms in both expressions coincide. The kinetic (geodetic) terms of Hamiltonians for geodetic and potential systems are based on $\widetilde{\Gamma}$-tensors, namely,

$$\mathcal{T} = \frac{1}{2}\widetilde{\Gamma}^{\mu\nu}(q)p_\mu p_\nu,$$

where

$$\widetilde{\Gamma}^{\mu\kappa}\Gamma_{\kappa\nu} = \delta^\mu{}_\nu$$

and

$$p_\mu = \frac{\partial L}{\partial \dot{q}^\mu} = \frac{\partial T}{\partial \dot{q}^\mu}$$

are canonical momenta conjugate to q^μ (dual objects to generalized velocities $\dot{q}^\mu$).

The above objects ω, θ and E, H possess natural generalizations to curved manifolds with affine connection. They appear there, respectively, as the connection form, canonical form, fundamental vector fields, and standard horizontal vector fields on $\mathrm{F}M$ (the principle fibre bundle of linear frames in a manifold M) [34]. Such a formalism is used in mechanics of infinitesimal affinely-rigid bodies, when affine degrees of freedom are considered as internal ones, i.e., attached to material points moving in manifolds with curvature and torsion [74].

Chapter 5

Without translational motion

It is instructive to consider the simplified situation when the centre of mass is at rest and the covariant translational forces do vanish, i.e, $Q_i = 0$. For the "usual" d'Alembert model such a situation, characteristic for practical elastic problems, was discussed detailly in our earlier papers [58, 62, 63, 65, 66, 67, 68]. And from the purely geometric symmetry point of view nothing particularly interesting happened there due to this simplification. In our model, based on dynamical affine symmetries, the translationless situation is an important step of the general analysis.

5.1 Equations of translationless motion

We have mentioned that in spatially affine $\mathcal{L}$-models with $Q_i = 0$, in particular in geodetic ones, canonical linear momentum is a constant of motion but translational velocity is not (except some very special solutions). This violates our ideas about Galileian symmetry, at least in the form developed in the "usual" d'Alembert mechanics. Nevertheless, the concept of translationless motion is well-defined because in the usual potential models equations $v^i = 0$, $p_i = 0$ are equivalent; this is one of exceptional cases when the constancy of velocity does not contradict the constancy of linear momentum. One must only remember that the Galilei transforms (in the

usual sense) of such space-resting solutions will not be solutions any longer.

In $\mathcal{L}$-models without translational motion the evolution is ruled by the second of the balance laws (4.11) with the simplified right-hand side, i.e.,

$$\frac{d\Sigma^i{}_j}{dt} = Q^i{}_j.$$

Affine invariance in M implies that in the completely geodetic case this becomes simply the Noether conservation law:

$$\frac{d\Sigma^i{}_j}{dt} = 0,$$

i.e., affine spin in spatial representation is a constant of motion. As mentioned, to obtained a closed system of equations, one must consider the above balance (conservation) jointly with the Legendre transformation and the definition of affine velocity. Unfortunately, the nice form (4.8) with constant coefficients cannot be used because $\hat{\Sigma}^A{}_B$ is not a constant of motion in the geodetic case. And in general, (4.14) is too complicated to be effectively used. But it turns out that something may be done for our simplified model (4.16), affine in M and η-metrical in U.

Something similar may be said about $\mathcal{R}$-models, moreover, they are in some respects simpler. The balance equations (4.13) reduce to their internal parts, i.e.,

$$\frac{d\hat{\Sigma}^A{}_B}{dt} = \hat{Q}^A{}_B,$$

and become conservation laws for the co-moving affine spin in the geodetic case, i.e.,

$$\frac{d\hat{\Sigma}^A{}_B}{dt} = 0.$$

As previously, the simplicity is only seeming one because the laboratory components $\Sigma^i{}_j$ fail to be constants of motion and Legendre transformation expressed in co-moving terms (4.15), in general, is rather complicated. Fortunately, for our models (4.17), metrical in space and centro-affine in the material, also something may be done.

5.2 General solution for doubly affine model

Let us begin with the over-simplified model with $I = 0$, affine both in the spatial and material sense. It is easily seen that the general solution for

translation-free geodetic motion is then given by the system of orbits of one-parameter subgroups of $\mathrm{GL}(V)$ or, equivalently, one-parameter subgroups of $\mathrm{GL}(U)$, i.e.,

$$\varphi(t) = \exp(Et)\varphi_0 = \varphi_0 \exp(\hat{E}t), \tag{5.1}$$

where φ_0 is an arbitrary element of $\mathrm{LI}(U, V)$, E is an arbitrary element of $\mathrm{L}(V) = \mathrm{GL}(V)'$, and $\hat{E} = \varphi_0^{-1} E \varphi_0$ is the corresponding element of $\mathrm{L}(U) = \mathrm{GL}(U)'$ obtained by the φ_0-similarity. If we identify formally U and V with $\mathbb{R}^n$ (by the particular choice of bases), then the phase portrait consists of all one-parameter subgroups of $\mathrm{GL}(n, \mathbb{R})$ and of all their left cosets or, equivalently, of all their right cosets. One must only remember that although the sets of left and right cosets coincide, they are parameterized in a different way by the corresponding generators and initial shifting elements. The reason is that $\mathrm{GL}(n, \mathbb{R})$ is non-Abelian and, in general, its one-parameter subgroups are not normal. If we write the group-theoretical version of (5.1), i.e.,

$$g(t) = \exp(at)h = h \exp(h^{-1}aht),$$

it is seen that the coinciding left and right cosets usually refer to different generators a and $h^{-1}ah$, thus, to different subgroups. If some left and right cosets refer to the same subgroup, i.e., the same generator a, and have non-empty intersection, then, as a rule, they are different subsets, i.e.,

$$g_1(t) = \exp(at)h \neq h \exp(at) = g_2(t).$$

Only the dilatational subgroup is exceptional because, consisting of central elements, it is a normal subgroup, and $h^{-1}ah = a$ for any dilatation generator a; the previous inequality becomes equality for any $h \in \mathrm{GL}(n, \mathbb{R})$.

Let us notice that in the solution (5.1) the pairs φ_0, E and $\varphi_0, \hat{E}$ play the role of differently represented initial conditions; in this sense they label the general solution. Thus, $\varphi_0 = \varphi(0)$ is an initial configuration, whereas $E = \Omega(0)$, $\hat{E} = \hat{\Omega}(0)$ are initial and at the same time permanently constant values of the laboratory and co-moving affine velocities. Therefore, the initial values of generalized velocities are given by $\dot{\varphi}(0) = E\varphi_0 = \varphi_0 \hat{E}$.

5.3 Stationary rotations

It is seen that for $I = 0$ the structure of general solution resembles that of the spherical rigid body. It is so for every geodetic model on a semisimple

group or its trivial central extension if the kinetic energy is doubly (left and right) invariant [2]. But we should remember that even in the simple case of a free anisotropic rigid body situation changes drastically. Kinetic energy is invariant under left regular translations on $\mathrm{SO}(3,\mathbb{R})$ (identified with the configuration space) but no longer under right translations. As a rule, one-parameter subgroups and their cosets fail to be solutions, i.e., they are not geodetics of left-invariant metric tensors on $\mathrm{SO}(3,\mathbb{R})$. There are some exceptions, however, namely stationary rotations [2, 44, 45, 53]. They happen when one of main axes of inertia has a fixed orientation in space and the remaining two perform a uniform rotation about it with a fixed angular velocity. Thus, there is a subset of general solution given by three one-parameter subgroups and all their left cosets (the non-moving axis of inertia may be arbitrarily oriented in space). This is the special case of what is known as relative equilibria [1, 44, 45]. They correspond to critical points of geodetic Hamiltonians restricted to co-adjoint orbits in the dual space of the Lie algebra $\mathrm{SO}(3,\mathbb{R})' \simeq \mathrm{SO}(3,\mathbb{R})$ [1, 44, 45]. Such particular solutions, although do not exhaust the phase portrait, contain an important information about its structure.

5.4 Generalized equilibria as special solutions of $\mathcal{L}$-models

Something similar happens in our affine models when $I \neq 0$. The general solution is not any longer given by orbits of one-parameter subgroups. Nevertheless, there exist geometrically interesting orbits which are particular solutions, i.e., generalized equilibria.

5.4.1 Left-cosets representation

Let us begin with geodetic $\mathcal{L}$-models (4.16) left-invariant under $\mathrm{GL}(V)$ and right-invariant under $\mathrm{O}(U,\eta)$. One can show after some calculations that there exist solutions of the following form:

$$\varphi(t) = \varphi_0 \exp(Ft), \tag{5.2}$$

where the initial configuration $\varphi_0 \in \mathrm{LI}(U,V)$ is arbitrary just as in (5.1). But now $F \in \mathrm{L}(U) \simeq \mathrm{GL}(U)'$ is not arbitrary any longer, instead it must

be η-normal in the sense of commuting with its η-transpose, i.e.,

$$F^A{}_C\eta^{CD}F^E{}_D\eta_{EB} - \eta^{AD}F^E{}_D\eta_{EC}F^C{}_B = 0.$$

Introducing the η-transpose symbol,

$$(F^{\eta T})^A{}_B := \eta^{AC}F^D{}_C\eta_{DB}, \tag{5.3}$$

we can write the above condition in the following concise form:

$$[F, F^{\eta T}] = FF^{\eta T} - F^{\eta T}F = 0. \tag{5.4}$$

It is obvious that for such solutions affine velocities are constant and given by

$$\Omega = \varphi_0 F \varphi_0^{-1}, \qquad \hat{\Omega} = F.$$

The initial data φ_0, F are independent of each other. The only restriction is that of η-normality imposed on F alone. This holds, in particular, in two extremely opposite special cases when F is η-skew-symmetric or η-symmetric, i.e.,

$$F^{\eta T} = -F, \qquad F^{\eta T} = F.$$

In the skew-symmetric case the one-parameter group generated by F consists of η-rotations, i.e.,

$$\exp(Ft) \in \mathrm{SO}(U, \eta) \subset \mathrm{GL}(U).$$

If F is η-symmetric, then so are transformations $\exp(Ft)$; they describe pure deformations in U in the sense of η-polar decomposition.

In calculations one identifies usually U and V with $\mathbb{R}^n$ and their metrics η, g with the Kronecker delta. Then the solutions (5.2) become all possible left cosets of one-parameter subgroups of $\mathrm{GL}(n, \mathbb{R})$ generated by all possible normal matrices $F \in \mathrm{L}(n, \mathbb{R})$, i.e.,

$$[F, F^T] = FF^T - F^TF = 0$$

(in this formula we mean the usual matrix transposition).

5.4.2 Right-cosets representation

Following (5.1) we can try to rewrite (5.2) in terms of the left-acting one-parameter subgroups. It is easy to see that

$$\varphi(t) = \varphi_0 \exp(Ft) = \exp(\check{F}t)\varphi_0, \tag{5.5}$$

where $\check{F} = \varphi_0 F \varphi_0^{-1} \in \mathrm{L}(V) = \mathrm{GL}(V)'$.

In this representation φ_0 is still arbitrary but $\check{F}$ is subject to some restrictions following from (5.4) and depending on φ_0. Namely, $\check{F}$ is normal in the sense of the Cauchy deformation tensor $C[\varphi_0]$ used as a kind of metric in V, i.e.,

$$\check{F}^i{}_a C[\varphi_0]^{ak} \check{F}^l{}_k C[\varphi_0]_{lj} - C[\varphi_0]^{ik} \check{F}^l{}_k C[\varphi_0]_{la} \check{F}^a{}_j = 0.$$

Introducing in analogy to (5.3) the $C[\varphi_0]$-transpose of $\check{F}$,

$$(\check{F}^{C[\varphi_0]T})^i{}_j := \widetilde{C}[\varphi_0]^{ik} \check{F}^l{}_k C[\varphi_0]_{lj},$$

we can write simply

$$[\check{F}, \check{F}^{C[\varphi_0]T}] = 0. \tag{5.6}$$

Therefore, in the right-cosets representation the initial configuration φ_0 and the generator $\check{F}$ are mutually interrelated. Namely, if φ_0 is not subject to any restrictions, then $\check{F}$ satisfies the condition (5.6) explicitly depending on φ_0. And conversely, if $\check{F}$ is arbitrary, then the initial conditions of φ_0 must be so suited to any particular choice of $\check{F}$ that the commutator condition (5.6) is non-violated.

5.5 Special solutions with fixed metric tensors

Let us observe that in all $\mathcal{L}$-models the spatial metric g does not occur in expressions for the kinetic energy at all. Thus, as a matter of fact it does not need to exist at all and the physical space M may be purely affine. Only the material metric η in the body is essential for (4.16). Let us notice, however, that if both $g \in V^* \otimes V^*$ and $\eta \in U^* \otimes U^*$ are fixed, then some family of special solutions may be distinguished, for which the relationship between initial configurations and infinitesimal generators is simpler and

more symmetric. Namely, we can start from the very beginning with the representation

$$\varphi(t) = \exp(Et)\varphi_0,$$

where E and φ_0 are respectively some elements of $\mathrm{L}(V)$ and $\mathrm{LI}(U, V)$. It is easy to see that, when some metric g is fixed in V, then we have at disposal a very natural family of solutions assuming that $\varphi_0 \in \mathrm{LI}(U, V)$ is an isometry and E is g-normal,

$$[E, E^{gT}] = EE^{gT} - E^{gT}E = 0, \tag{5.7}$$

where in a full analogy to the previous expression we use the definition

$$\left(E^{gT}\right)^i{}_j := g^{ik}E^l{}_k g_{lj}.$$

Obviously, such solutions form a submanifold of the family (5.5), (5.6) because then $C[\varphi_0] = g$.

5.6 Generalized equilibria as special solutions of $\mathcal{R}$-models

Now let us consider geodetic $\mathcal{R}$-models (4.17), which are left-invariant only under $\mathrm{O}(V, g)$ but right-invariant under the total $\mathrm{GL}(U)$. Now, as expected, the situation will be reversed. Let us assume solutions in the right-coset form:

$$\varphi(t) = \exp(Et)\varphi_0,$$

where $\varphi_0 \in \mathrm{LI}(U, V)$, $E \in \mathrm{L}(V)$. It is easy to show that such a curve (right coset) satisfies, in fact, equations of geodetic motion if E is g-normal, just as in (5.7), but now φ_0 may be quite arbitrary isomorphism of U onto V. And if we write the above curve as a left coset, i.e.,

$$\varphi(t) = \varphi_0 \exp(\widetilde{E}t), \qquad \widetilde{E} = \varphi_0^{-1}E\varphi_0 \in \mathrm{L}(U),$$

then it is easy to see that, with a still arbitrary φ_0, $\widetilde{E}$ will be $G[\varphi_0]$-normal in the sense of Green deformation tensor $G[\varphi_0] \in U^* \otimes U^*$, i.e.,

$$[\widetilde{E}, \widetilde{E}^{G[\varphi_0]T}] = \widetilde{E}\widetilde{E}^{G[\varphi_0]T} - \widetilde{E}^{G[\varphi_0]T}\widetilde{E} = 0,$$

where the $G[\varphi_0]$-transpose is defined in a full analogy to the above η- and g-transposes,

$$(\widetilde{E}^{G[\varphi_0]T})^A{}_B := G[\varphi_0]^{AC}\widetilde{E}^D{}_C G[\varphi_0]_{DB}.$$

5.7 Special solutions for fixed metric tensors

Just as previously, we can distinguish an interesting submanifold of such solutions when some material metric tensor $\eta \in U^* \otimes U^*$ is fixed (we know it does not exist in (4.17)). They are given by curves of the following form:

$$\varphi(t) = \varphi_0 \exp(Ft),$$

where $\varphi_0 \in \mathrm{O}(U, \eta; V, g)$ is an isometry and $F \in \mathrm{L}(U)$ is η-normal in the sense of (5.3), (5.4). For such solutions we have $G[\varphi_0] = \eta$. Manipulating with η we introduce some kind of parametrization, ordering in our manifold of relative equilibria.

5.8 Volume and dilatation measures

The above particular solutions are very special, nevertheless very important. Their position in our model is analogous to that of stationary rotations in rigid body mechanics. They provide a kind of skeleton for the general solution. Nevertheless, some, at least qualitative, rough knowledge of the phase portrait would be mostly welcome. The crucial question is to what extent the purely geodetic models may predict bounded motions. Obviously, this is impossible for compressible bodies, when the configuration space is identical with the total $\mathrm{LI}(U, V)$. To see this it is sufficient to consider the special case $n = 1$, when compressibility is the only degree of freedom of internal motion. There is only one affinely-invariant model of T_{int}. The resulting trivial geodetic model predicts, depending on the sign of the initial internal velocity, either the infinite expansion or contraction, although in the latter case the object shrinks to a single point after infinite time. The only bounded (and non-stable) solution is the rest state. Something similar occurs in n-dimensional geodetic problems. Namely, degrees of freedom of the isochoric motion are orthogonal to the pure dilatations and completely independent of them. Some purely geometric comments are necessary here. Namely, if N and M are purely amorphous affine spaces, in particular no metrics η, g are fixed in U, V, then their volume measures are defined only up to multiplicative constant factors. They are Lebesgue measures, i.e., special cases of Haar measures invariant under additive Abelian translations in U, V (in N, M). Let us denote some particular choices respectively by ν_U, ν_V. Obviously, for any measurable domain $Y \subset U$ and for any configuration

$\varphi \in \mathrm{LI}(U,V)$ we have

$$\nu_V(\varphi(Y)) = \Delta(\varphi)\nu_U(Y).$$

The scalar multiplicator $\Delta(\varphi)$ depends on φ and on non-correlated normalizations of ν_V, ν_U but does not depend on Y. Obviously, for any $A \in \mathrm{GL}(U)$, $B \in \mathrm{GL}(V)$ we have:

$$\Delta(A\varphi B) = (\det A)\Delta(\varphi)\det B.$$

The motion is isochoric if Δ is constant in the course of evolution. Obviously, this definition is independent of particular normalizations of ν_V, ν_U. The manifold $\mathrm{LI}(U,V)$ becomes then foliated by (n^2-1)-dimensional leaves consisting of mutually non-compressed configurations. Every such leaf establishes holonomic constraints, and the total foliation is what is sometimes referred to as semi-holonomic or quasi-holonomic constraints. If some metric tensors $\eta \in U^* \otimes U^*$, $g \in V^* \otimes V^*$ are fixed, then the measures ν_U, ν_V may be fixed respectively as ν_η, ν_g, and in terms of coordinates

$$d\nu_\eta = \sqrt{\det[\eta_{KL}]}da^1 \cdots da^n, \qquad d\nu_g = \sqrt{\det[g_{ij}]}dx^1 \cdots dx^n.$$

Using Euclidean coordinates we can simply put

$$\Delta(\varphi) = \det[\varphi^i{}_K]$$

but, obviously, this convention fails for general coordinates. For non-Euclidean affine coordinates we have

$$\Delta(\varphi) = \frac{\sqrt{\det[g_{kl}]}}{\sqrt{\det[\eta_{AB}]}}\det[\varphi^i{}_K].$$

Let us remind that the corresponding curvilinear formula reads [19]:

$$\frac{d\nu_g(x(a))}{d\nu_\eta(a)} = \frac{\sqrt{\det[g_{kl}(x(a))]}}{\sqrt{\det[\eta_{AB}(a)]}}\det\left[\frac{\partial x^i}{\partial a^K}\right]. \tag{5.8}$$

If some volumes are fixed in U and V, e.g., due to some choices of metrics η, g, then the volume extension ratio $\Delta(\varphi)$ is uniquely fixed. In certain formulas it may be convenient to use the additive parameter $\alpha(\varphi)$ instead of the multiplicative one,

$$\Delta(\varphi) = \exp\left[\alpha(\varphi)\right].$$

Another convenient dilatation measures are those describing the linear size extension ratio,

$$D(\varphi) = \sqrt[n]{\Delta(\varphi)} = \exp\left[\frac{\alpha(\varphi)}{n}\right] = \exp\left[q(\varphi)\right].$$

5.9 Stabilizing of dilatations

The only possibility of stabilizing dilatations is to include some extra potential preventing the unlimited expansion to the infinite size and asymptotic contraction to the point-like object. There is plenty of such phenomenological modelling potentials, e.g.,

$$V_{\rm dil} = \frac{\kappa}{8}\left(D^2 + D^{-2} - 2\right) = \frac{\kappa}{8}\left({\rm ch}2q - 1\right), \qquad \kappa > 0.$$

Obviously, this potential is positively infinite at $q = \mp\infty$ ($D = 0$, $D = +\infty$) and has the global stable equilibrium at $q = 0$ ($D = 1$), where it behaves as the harmonic oscillator: $V_{\rm dil}(q) \approx \kappa q^2/2$ for $q \approx 0$. For strongly extended bodies it also behaves harmonically in the D-variable sense. Another phenomenological model would be just the global form $\kappa q^2/2$. One can also try to use some toy models predicting "dissociation" of the body (its unlimited size-expansion), unlimited collapse, or both of them above some threshold of the total dilatational energy, e.g.,

$$V_{\rm dil}(q) = \frac{\kappa}{2}\left({\rm th}^2 q - 1\right).$$

In certain problems it may be reasonable to use phenomenological models preventing contraction but admitting dissociation.

In quantized version of the theory one can stabilize dilatations in an easy way with the use of the q-variable potential well (perhaps with the infinite walls) concentrated around $q = 0$ ($D = 1$).

5.10 Analytical identification

If we identify analytically U and V with $\mathbb{R}^n$ and ${\rm LI}(U, V)$ with ${\rm GL}(n, \mathbb{R})$, then it is clear that the connected component of unity ${\rm GL}^+(n, \mathbb{R})$ becomes the direct product ${\rm GL}^+(n, \mathbb{R}) \simeq {\rm SL}(n, \mathbb{R}) \times \exp(\mathbb{R}) = {\rm SL}(n, \mathbb{R}) \times \mathbb{R}^+$; the

second group factor is obviously meant in the multiplicative sense, i.e., as $\mathrm{GL}^+(1,\mathbb{R})$. It describes pure dilatations, whereas $\mathrm{SL}(n,\mathbb{R})$ refers to the isochoric motion. Without this identification, $\mathrm{LI}(U,V)$ may be represented as the Cartesian product of any of the aforementioned leaves (of mutually non-compressed configurations) and the multiplicative group $\mathbb{R}\backslash\{0\}$. If some volume-standards ν_U, ν_V (e.g., metric-based ones ν_η, ν_g) and orientations are fixed in U, V, then $\mathrm{LI}^+(U,V)$, i.e., the manifold of orientation-preserving isomorphisms, is identified with the product $\mathrm{SLI}(U,\nu_U;V,\nu_V)\times\exp(\mathbb{R})$, where, obviously, the first term consists of transformations φ for which $\Delta(\varphi)=1$, i.e., $q(\varphi)=0$. Such a formulation is more correct from the point of view of geometrical purity, however, for our purposes (qualitative discussion of the general solution), the analytical identification of $\mathrm{LI}^+(U,V)$ with $\mathrm{GL}^+(n,\mathbb{R})\simeq\mathrm{SL}(n,\mathbb{R})\times\exp(\mathbb{R})$ is sufficient and, as a matter of fact, more convenient. In any case, qualitative analysis of the general solution (bounded and non-bounded trajectories) is not then obscured by cosmetical aspects of geometry. Thus, from now on the internal configuration space $Q_{\mathrm{int}}=\mathrm{LI}(U,V)$ will be identified with $\mathrm{GL}^+(n,\mathbb{R})\simeq\mathrm{SL}(n,\mathbb{R})\times\exp(\mathbb{R})$. Any matrix $\varphi\in\mathrm{GL}^+(n,\mathbb{R})$ will be uniquely represented as follows:

$$\varphi = l\Psi = \exp(q)\Psi, \qquad \Psi\in\mathrm{SL}(n,\mathbb{R}).$$

It is convenient to introduce the following shear velocities:

$$\omega := \frac{d\Psi}{dt}\Psi^{-1}, \qquad \hat{\omega} := \Psi^{-1}\frac{d\Psi}{dt}.$$

Obviously, ω, $\hat{\omega}\in\mathrm{SL}(n,\mathbb{R})'$, i.e., they are trace-less. Then affine velocities may be expressed as follows:

$$\Omega = \omega + \frac{dq}{dt}I, \qquad \hat{\Omega} = \hat{\omega} + \frac{dq}{dt}I,$$

where, obviously, I denotes the identity matrix.

Analogously, the affine spin splits as follows:

$$\Sigma = \sigma + \frac{p}{n}I, \qquad \hat{\Sigma} = \hat{\sigma} + \frac{p}{n}I, \qquad \sigma,\ \hat{\sigma}\in\mathrm{SL}(n,\mathbb{R})',$$

where p denotes the dilatational canonical momentum. The velocity-momentum pairing becomes:

$$\mathrm{Tr}(\Sigma\Omega) = \mathrm{Tr}(\hat{\Sigma}\hat{\Omega}) = \mathrm{Tr}(\sigma\omega) + p\dot{q} = \mathrm{Tr}(\hat{\sigma}\hat{\omega}) + p\dot{q}.$$

Poisson-bracket relations for σ-components are based on the structure constants of $\mathrm{SL}(n,\mathbb{R})$. The same is obviously true for $\hat{\sigma}$ with the proviso that the signs are reversed. The mixed $\{\sigma,\hat{\sigma}\}$ brackets do vanish. Obviously, $\{q,p\}=1$, and the quantities q, p (dilatation) Poisson-commute with Ψ, σ, $\hat{\sigma}$ (shear).

5.11 Isochoric and dilatational sectors

The doubly-invariant "kinetic energy" (4.3) is a superposition of the isochoric and dilatational terms,

$$T=\frac{A}{2}\mathrm{Tr}(\omega^2)+\frac{n(A+nB)}{2}\dot{q}^2=T_{\mathrm{sh}}+T_{\mathrm{dil}}.$$

Performing the Legendre transformation,

$$\sigma=A\omega,\qquad p=n(A+nB)\dot{q},$$

we obtain the following geodetic Hamiltonian:

$$\mathcal{T}=\frac{1}{2A}\mathrm{Tr}(\sigma^2)+\frac{1}{2n(A+nB)}p^2=\mathcal{T}_{\mathrm{sh}}+\mathcal{T}_{\mathrm{dil}}. \tag{5.9}$$

In these expressions the quantities ω, σ may be replaced by their co-moving representations $\hat{\omega}$, $\hat{\sigma}$. Lagrangians and Hamiltonians of systems with stabilized (controlled) dilatations have the form:

$$L=L_{\mathrm{sh}}+L_{\mathrm{dil}}=T_{\mathrm{sh}}+T_{\mathrm{dil}}-V(q),$$

$$H=H_{\mathrm{sh}}+H_{\mathrm{dil}}=\mathcal{T}_{\mathrm{sh}}+\mathcal{T}_{\mathrm{dil}}+V(q).$$

There is a complete separability of shear and dilatation degrees of freedom; they are mutually independent. This property would not be violated if we included also a shear potential $V_{\mathrm{sh}}(\Psi)$, i.e., if $V(\Psi,q)=V_{\mathrm{sh}}(\Psi)+V_{\mathrm{dil}}(q)$. The question arises as to the structure of general solution for the geodetic $\mathrm{SL}(n,\mathbb{R})$-model, i.e., for $V_{\mathrm{sh}}=0$. A superficial reasoning based on the analogy with d'Alembert models might suggest that the general solution consists only of unbounded motions (and the rest states), because there is no potential and the configuration space is non-compact. However, it is not the case; there is an open subset consisting of bounded orbits.

5.12 Qualitative analysis of solutions

Indeed, let us assume that some trace-less matrix $\alpha \in \mathrm{SL}(n,\mathbb{R})'$ is similar to an anti-symmetric matrix $\lambda \in \mathrm{SO}(n,\mathbb{R})'$, i.e., there exists such $\chi \in \mathrm{SL}(n,\mathbb{R})$ that $\alpha = \chi\lambda\chi^{-1}$. Then every motion $\Psi(t) = \exp(\alpha t)\Psi_0$ is bounded. Indeed, $\exp(\lambda t)$ is a bounded subgroup of $\mathrm{SO}(n,\mathbb{R}) \subset \mathrm{SL}(n,\mathbb{R})$ and so is $\exp(\alpha t) = \chi\exp(\lambda t)\chi^{-1}$. But similarities are globally defined continuous mappings, therefore, they transform bounded subsets onto bounded ones. Let us observe that for physical dimensions $n = 2, 3$ motions of this type are periodic. For higher dimensions periodicity is not necessary, although obviously possible. To see this, let us consider the simplest situation $n = 4$ and represent $\mathbb{R}^4$ as the direct sum of two complementary $\mathbb{R}^2$-subspaces. Now, let $\lambda \in \mathrm{SO}(n,\mathbb{R})'$ be a block matrix consisting of two skew-symmetric blocks $\nu_1 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\nu_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Rotations generated by separate blocks are obviously periodic, but the total motion is periodic if and only if $\nu_1/\nu_2 \in \mathbb{Q}$, i.e., the ratio of angular velocities is a rational number. If it is irrational, the subgroup obtained by exponentiation of t-multiples of the above block matrix is not a periodic function of the parameter t. It is not a closed subset at all; its closure is a two-dimensional submanifold of $\mathrm{SO}(4,\mathbb{R}) \subset \mathrm{SL}(4,\mathbb{R})$. Therefore, being an algebraic subgroup of $\mathrm{SO}(4,\mathbb{R}) \subset \mathrm{SL}(4,\mathbb{R})$ it is not its Lie subgroup in the literal sense. The same concerns any subgroup of $\mathrm{SL}(4,\mathbb{R})$ obtained from the above one by a similarity transformation. For an arbitrary n, solutions of this kind are matrix-valued almost periodic functions of the time variable t.

5.13 Unbounded motion. Stability problem

If $\alpha \in \mathrm{SL}(n,\mathbb{R})'$ is similar to a symmetric matrix $\kappa \in \mathrm{SO}(n,\mathbb{R})'$, $\alpha = \chi\kappa\chi^{-1}$, then, obviously, the motion given by $\Psi(t) = \exp(\alpha t)\Psi_0$ is unbounded. But one can show that the previously described bounded almost periodic solutions are "stable" in such a sense that for any skew-symmetric λ there exists some open range of symmetric κ-s such that for $\alpha = \lambda + \kappa$, or, more generally, for similar matrices $\alpha = \chi(\lambda + \kappa)\chi^{-1}$, the corresponding solutions $\Psi(t) = \exp(\alpha t)\Psi_0$ are also bounded, although not necessarily almost periodic [54, 89] (and not necessarily periodic in dimensions $n = 2, 3$). The arbitrariness of pairs (α, Ψ_0) is sufficient for the corresponding fam-

ily of bounded solutions to be open in the general solution manifold, thus, $2(n^2-1)$-dimensional (topological and differential concepts like openness and dimension are meant in the sense of the manifold of initial conditions). Let us observe that this statement would be false for solutions with generators α similar to skew-symmetric matrices. At first look, this might seem strange, because the structure of $\mathrm{SL}(n,\mathbb{R})'$ implies that the family of such α-s is (n^2-1)-dimensional and so is the set of initial configurations Ψ_0. But these data are not independent and mutually interfere in the formula $\Psi(t)=\exp(\alpha t)\Psi_0$. Therefore, the very interesting subfamily of almost-periodic solutions is a proper subset of the manifold of all bounded solutions.

By "anti-analogy", for symmetric matrices $\lambda=\lambda^T\in\mathrm{SL}(n,\mathbb{R})'$ the corresponding solutions $\Psi(t)=\exp(\lambda t)\Psi_0$ are non-bounded (escaping in $\mathrm{SL}(n,\mathbb{R})$) and, obviously, so are the solutions generated by matrices similar to symmetric ones, $\Psi(t)=\chi\exp(\lambda t)\chi^{-1}\Psi_0=\exp(\chi\lambda\chi^{-1}t)\Psi_0$. And again this property is stable with respect to small perturbations of λ by skew-symmetric matrices $\epsilon=-\epsilon^T\in\mathrm{SO}(n,\mathbb{R})'$. Therefore, the general solution of the geodetic doubly-invariant model contains also an open subset of non-bounded (escaping) solutions.

5.14 Remarks

Roughly speaking, using analogy with the Kepler or attractive Coulomb problem we may tell here about motions below and above dissociation threshold, however, without any potential, just in purely geodetic models on the non-compact manifold $\mathrm{SL}(n,\mathbb{R})$.

In a quantized version of this model the family of bounded classical trajectories is replaced by the discrete energy spectrum and the L^2-class wave functions of stationary states. And similarly, the manifold of non-bounded orbits is a classical counterpart of the continuous spectrum and non-normalized wave functions (scattering situations).

Obviously, the above description in terms of groups $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{R})$ is an analytical simplification used for computational purposes. To use systematically a more correct geometrical language we should replace the terms "skew-symmetric" and "symmetric" by g- or η-skew-symmetric and symmetric:

$$\lambda^i{}_j=\mp g^{ik}\lambda^m{}_k g_{mj},\qquad \hat\lambda^A{}_B=\mp\eta^{AC}\hat\lambda^D{}_C\eta_{DB}.$$

Finally, we can conclude that if dilatations are stabilized by some potential $V_{\rm dil}(q)$, then for the model with the kinetic energy invariant under spatial and material affine transformations, there exists a $2n^2$-dimensional family of bounded solutions even if the shear component of motion is purely geodetic. If the stabilizing dilatational potential has an upper bound, there exists also a $2n^2$-dimensional family of unbounded, escaping motions. The above arguments are based on properties of one-parameter subgroups and their cosets in $\mathrm{SL}(n,\mathbb{R})$. Therefore, they do not apply directly to affine-metrical and metrical-affine models. Indeed, as we have seen, if the spatial or material symmetry of the kinetic energy is restricted to the rotation group, then, except some special solutions (relative equilibria), one-parameter subgroups and their cosets fail to be solutions. Nevertheless, our arguments may be used then in a non-direct way.

5.15 Affine-metrical and metrical-affine models

In analogy to (5.9) we can rewrite the kinetic Hamiltonians (4.26), (4.27), i.e., (4.30), (4.31), as follows:

$$\begin{aligned}
\mathcal{T}_{\rm int} &= \frac{1}{2(I+A)}\mathrm{Tr}(\hat{\sigma}^2)+\frac{1}{2n(I+A+nB)}p^2+\frac{I}{2(I^2-A^2)}\|V\|^2 \qquad (5.10)\\
&= \frac{1}{2(I+A)}C_{\mathrm{SL}(n)}(2)+\frac{1}{2n(I+A+nB)}p^2+\frac{I}{2(I^2-A^2)}\|V\|^2,\\
\mathcal{T}_{\rm int} &= \frac{1}{2(I+A)}\mathrm{Tr}(\sigma^2)+\frac{1}{2n(I+A+nB)}p^2+\frac{I}{2(I^2-A^2)}\|S\|^2 \qquad (5.11)\\
&= \frac{1}{2(I+A)}C_{\mathrm{SL}(n)}(2)+\frac{1}{2n(I+A+nB)}p^2+\frac{I}{2(I^2-A^2)}\|S\|^2,
\end{aligned}$$

where $C_{\mathrm{SL}(n)}(2)=\mathrm{Tr}(\sigma^2)=\mathrm{Tr}(\hat{\sigma}^2)$.

The formulas (4.30), (4.31) or, equivalently, (5.10), (5.11) imply that for dilatationally-stabilized models $H=\mathcal{T}_{\rm int}+V_{\rm dil}(q)$ with the affine-metrical and metrical-affine kinetic terms $\mathcal{T}_{\rm int}$, all the above statements concerning bounded and unbounded solutions of affine-affine models (5.9), (4.3) remain true. In particular, for the purely geodetic incompressible models with $\mathcal{T}_{\rm int}$ invariant under $\mathrm{SL}(V)\times\mathrm{O}(U,\eta)$ or under $\mathrm{O}(V,g)\times\mathrm{SL}(U)$, there exists an open subset of bounded solutions (vibrations) and an open subset of

non-bounded ones. What concerns spatially affine and materially metrical models, the very rough argument is that the evolution of quantities Σ, $\mathcal{K}$ is exactly the same as it was for Hamiltonians H with $\mathcal{T}_{\rm int}$ affinely-invariant both in the physical and in the material spaces, in this case in (5.9) A is replaced by $I + A$. This is a direct consequence of equations of motion written in terms of Poisson brackets,

$$\frac{dF}{dt} = \{F, H\}.$$

In fact, $\|V\|$ is a constant of motion for both types of Hamiltonians (affine-affine and affine-metrical). In addition to the Lie-algebraic relations of $\mathrm{GL}(V)' \simeq \mathrm{L}(V)$ satisfied by $\Sigma^i{}_j$, we have the following obvious Poisson rules:

$$\{\Sigma^i{}_j, C(2)\} = \{\Sigma^i{}_j, C(1)\} = 0,$$
$$\{\Sigma^i{}_j, \|V\|^2\} = 0, \qquad \{\mathcal{K}_a, \|V\|^2\} = 0.$$

The first equations express an obvious property of $C(k)$ as Casimir invariants of $\Sigma^i{}_j$ (and $\hat{\Sigma}^A{}_B$). The second formula follows from the obvious relationship $\{\Sigma^i{}_j, \hat{\Sigma}^A{}_B\} = 0$, because $\|V\|^2$ is an algebraic function of $\hat{\Sigma}^A{}_B$. And the third equation is due to the fact that the deformation invariants $\mathcal{K}_a$ are invariant under the group of material isometries generated by $V^A{}_B$.

Therefore, the time evolution of variables $\Sigma^i{}_j$, $\mathcal{K}_a$ is identical in both types of models, i.e., (5.9) and (5.10); the former with A replaced by $I + A$. As a matter of fact, for geodetic models with dilatation-stabilizing potentials $V(q)$, the deviator $\sigma^i{}_j = \Sigma^i{}_j - (1/n)\Sigma^a{}_a\delta^i{}_j$ is a constant of motion and, obviously, it is so for the purely geodetic incompressible models. The only difference occurs in degrees of freedom ruled by $\mathrm{SO}(V, g)$, $\mathrm{SO}(U, \eta)$, describing the orientation of principal axes of deformation tensors $C \in V^* \otimes V^*$, $G \in U^* \otimes U^*$. But, roughly speaking, these degrees of freedom have compact topology and their evolution does not influence the bounded or non-bounded character of the total orbits.

The same reasoning applies to dilatationally stabilized geodetic models invariant under $\mathrm{O}(V, g) \times \mathrm{GL}(U)$ or purely geodetic incompressible models invariant under $\mathrm{O}(V, g) \times \mathrm{SL}(U)$ (spatially metrical and materially affine models). Then everything follows from Poisson brackets:

$$\{\hat{\Sigma}^A{}_B, C(2)\} = \{\hat{\Sigma}^A{}_B, C(1)\} = 0,$$
$$\{\hat{\Sigma}^A{}_B, \|S\|^2\} = 0, \qquad \{\mathcal{K}_a, \|S\|^2\} = 0.$$

Now on the level of state variables $\hat{\Sigma}^A{}_B$, $\mathcal{K}_a$ the time evolution is exactly identical with that based on the affine-affine model of $\mathcal{T}_{\text{int}}$ (again with A in (5.9) replaced by $I + A$).

Let us stress an important point that it is the time evolution of deformation invariants that decides whether the total motion is bounded or not. This is a purely geometric fact independent on any particular dynamical model. There is an analogy with the material point motion in $\mathbb{R}^n$. An orbit is bounded if and only if the range of the radial variable r is bounded.

The above point plays an essential role in the qualitative discussion of deformative motion. It suggests one to use analytical descriptions of degrees of freedom based on deformation invariants.

Chapter 6

Analytical description

In chapter 3 some fundamental facts concerning deformation tensors and deformation invariants were summarized. Below we continue this subject and present some natural descriptions of affine degrees of freedom well-adapted to the study of isotropic problems.

6.1 Natural definitions

The material and physical spaces are endowed with fixed metric tensors, $\eta \in U^* \otimes U^*$, $g \in V^* \otimes V^*$, and any configuration $\varphi \in \mathrm{LI}(U,V)$ gives rise to the symmetric positively definite tensors $G[\varphi] \in U^* \otimes U^*$, $C[\varphi] \in V^* \otimes V^*$, i.e., Green and Cauchy deformation tensors. Raising their first indices respectively with the help of η and g, we obtain the mixed tensors $\hat{G}[\varphi] \in U \otimes U^*$, $\hat{C}[\varphi] \in V \otimes V^*$ with eigenvalues λ_a, λ_a^{-1}, $a = \overline{1,n}$. It is also convenient to use the quantities Q^a, q^a, where

$$Q^a = \exp(q^a) = \sqrt{\lambda_a}.$$

The diagonal matrix $D = \mathrm{diag}(Q^1, \ldots, Q^n)$ is identified with the linear mapping $D : \mathbb{R}^n \to \mathbb{R}^n$.

The configuration $\varphi \in \mathrm{LI}(U,V)$ may be characterized by D, i.e., by the system of fundamental stretchings $Q^a = \exp(q^a)$, and by the systems of eigenvectors $R_a \in U$, $L_a \in V$ of $\hat{G}$, $\hat{C}$ normalized, respectively, in the sense

of η and g,

$$\hat{G}R_a = \lambda_a R_a = \exp(2q^a)R_a, \qquad \hat{C}L_a = \lambda_a^{-1}L_a = \exp(-2q^a)L_a.$$

Obviously, when the spectrum is non-degenerate, then R_a, L_a are uniquely defined (up to re-ordering) and pair-wise orthogonal,

$$\eta(R_a, R_b) = \eta_{CD}R^C{}_aR^D{}_b = \delta_{ab} = g(L_a, L_a) = g_{ij}L^i{}_aL^j{}_b.$$

Such a situation is generic, thus, when at some time instant $t \in \mathbb{R}$ $\varphi(t)$ corresponds to degenerate situation, then $L_a(t)$, $R_a(t)$ may be also uniquely defined due to the continuity demand.

The elements of the corresponding dual bases will be denoted respectively by $R^a \in U^*$, $L^a \in V^*$. When necessary, to avoid misunderstandings, we shall indicate explicitly the dependence of the above quantities on $\varphi \in \mathrm{LI}(U, V)$: $q^a[\varphi]$, $R_a[\varphi]$, $L_a[\varphi]$, etc.

Green and Cauchy deformation tensors may be respectively expressed as follows:

$$G[\varphi] = \sum_a \lambda_a[\varphi]R^a[\varphi] \otimes R^a[\varphi] = \sum_a \exp\left(2q^a[\varphi]\right) R^a[\varphi] \otimes R^a[\varphi],$$

$$C[\varphi] = \sum_a \lambda_a^{-1}[\varphi]L^a[\varphi] \otimes L^a[\varphi] = \sum_a \exp\left(-2q^a[\varphi]\right) L^a[\varphi] \otimes L^a[\varphi].$$

In this way φ has been identified with the triple of fictitious objects: two rigid bodies in U and V with configurations represented, respectively, by orthonormal frames $R \in \mathrm{F}(U, \eta)$, $L \in \mathrm{F}(V, g)$ and a one-dimensional n-particle system with coordinates q^a (or Q^a). Even for non-degenerate spectra of $\hat{G}[\varphi]$, $\hat{C}[\varphi]$ this representation is not unique because the labels a under the summation signs may be simultaneously permuted without affecting φ itself. For degenerate spectra this representation becomes continuously non-unique in a similar (although much stronger) way as, e.g., spherical coordinates at $r = 0$.

6.2 Two-polar decomposition

Let us observe that the linear frames $L = (\ldots, L_a, \ldots)$ and $R = (\ldots, R_a, \ldots)$ may be, as usual, identified with linear isomorphisms $L : \mathbb{R}^n \to V$ and

$R : \mathbb{R}^n \to U$. Similarly, their dual co-frames $\widetilde{L} = (\ldots, L^a, \ldots)$ and $\widetilde{R} = (\ldots, R^a, \ldots)$ are equivalent to isomorphisms $L^{-1} : V \to \mathbb{R}^n$ and $R^{-1} : U \to \mathbb{R}^n$. Identifying the diagonal matrix $\mathrm{diag}(\ldots, Q_a, \ldots)$ with a linear isomorphism $D : \mathbb{R}^n \to \mathbb{R}^n$, we may finally represent

$$\varphi = LDR^{-1},$$

this is a geometric description of what is sometimes referred to as the two-polar decomposition [66, 68, 70, 71, 72, 81].

Strictly speaking, in continuum mechanics, when the orientation of the body is constant during any admissible motion (no mirror-reflections), one has to fix some pattern orientations in U, V and admit only orientation-preserving mappings φ. And then the non-connected sets of all orthonormal frames $\mathrm{F}(U, \eta)$, $\mathrm{F}(V, g)$ are to be replaced by their connected submanifolds $\mathrm{F}^+(U, \eta)$, $\mathrm{F}^+(V, g)$ of positively oriented frames.

Obviously, the spatial and material orientation-preserving isometries $A \in \mathrm{SO}(V, g)$, $B \in \mathrm{SO}(U, \eta)$ affect only the L- and R-gyroscopes on the left. Indeed, $L \mapsto AL$, $R \mapsto BR$ result in

$$\varphi \mapsto A\varphi B^{-1}.$$

Their Hamiltonian generators, spin and minus-vorticity (i.e., respectively V- and U-spatial canonical spins) have identical Poisson-commutation rules.

For any of the mentioned rigid bodies, one can define in the usual way the angular velocity in two representations. One should stress that both V and U are from this point of view interpreted as "physical spaces". The "material" ones are both identified with $\mathbb{R}^n$. The "co-moving" and "current" representations $\hat{\chi} \in \mathrm{SO}(n, \mathbb{R})'$, $\chi \in \mathrm{SO}(V, g)'$ for the L-top are respectively given by

$$\hat{\chi}^a{}_b :=< L^a, \frac{dL_b}{dt} >= L^a{}_i \frac{dL^i{}_b}{dt},$$

$$\chi := \hat{\chi}^a{}_b L_a \otimes L^b, \text{ i.e., } \chi^i{}_j = \frac{dL^i{}_a}{dt} L^a{}_j.$$

The corresponding objects $\hat{\vartheta} \in \mathrm{SO}(n, \mathbb{R})'$, $\vartheta \in \mathrm{SO}(U, \eta)'$ for the R-top are defined by analogous formulas:

$$\hat{\vartheta}^a{}_b :=< R^a, \frac{dR_b}{dt} >= R^a{}_K \frac{dR^K{}_b}{dt},$$

$$\vartheta := \hat{\vartheta}^a{}_b R_a \otimes R^b, \text{ i.e., } \vartheta^K{}_L = \frac{dR^K{}_a}{dt} R^a{}_L.$$

In certain problems it is convenient to use non-holonomic velocities $\dot{q}^a$, $\hat{\chi}^a{}_b$, $\hat{\vartheta}^a{}_b$ or $\dot{q}^a$, $\chi^i{}_j$, $\vartheta^A{}_B$. Similarly, non-holonomic conjugate momenta p_a, $\hat{\rho}^a{}_b$, $\hat{\tau}^a{}_b$ or p_a, $\rho^i{}_j$, $\tau^A{}_B$ are used, where again

$$\hat{\rho}, \hat{\tau} \in \mathrm{SO}(n, \mathbb{R})', \qquad \rho \in \mathrm{SO}(V, g)', \quad \tau \in \mathrm{SO}(U, \eta)'. \tag{6.1}$$

The pairing between non-holonomic momenta and velocities is given by

$$< (\rho, \tau, p), (\chi, \vartheta, \dot{q}) > = < (\hat{\rho}, \hat{\tau}, p), (\hat{\chi}, \hat{\vartheta}, \dot{q}) >$$

$$= p_a \dot{q}^a + \frac{1}{2}\mathrm{Tr}(\rho\chi) + \frac{1}{2}\mathrm{Tr}(\tau\vartheta) = p_a \dot{q}^a + \frac{1}{2}\mathrm{Tr}(\hat{\rho}\hat{\chi}) + \frac{1}{2}\mathrm{Tr}(\hat{\tau}\hat{\vartheta}).$$

Remark: Our system of notations is slightly redundant, because ρ and τ coincide, respectively, with spin and negative vorticity,

$$\rho = S, \qquad \tau = -V. \tag{6.2}$$

The reason is that they are Hamiltonian generators of transformations

$$\varphi \mapsto A\varphi, \quad \varphi \mapsto \varphi B^{-1}, \quad A \in \mathrm{SO}(V, g), \quad B \in \mathrm{SO}(U, \eta). \tag{6.3}$$

The objects $\hat{\rho}$, $\hat{\tau}$ generate transformations

$$L \mapsto LA, \quad R \mapsto RB, \quad A, B \in \mathrm{SO}(n, \mathbb{R}) \tag{6.4}$$

and express the quantities ρ, τ in terms of the reference frames given, respectively, by the principal axes of the Cauchy and Green deformation tensors,

$$\rho = \hat{\rho}^a{}_b L_a \otimes L^b, \qquad \tau = \hat{\tau}^a{}_b R_a \otimes R^b.$$

Remark: In dynamical models based on the d'Alembert principle the quantities Q^a and their conjugate momenta P_a are more convenient that q^a and p_a. The latter ones are useful in models with affinely-invariant kinetic energy.

6.3 Polar decomposition

If V and U both are identified with $\mathbb{R}^n$ and $\mathrm{LI}(U,V)$ with $\mathrm{GL}(n,\mathbb{R})$, then L and R in the two-polar splitting $\varphi = LDR^{-1}$ become elements of $\mathrm{SO}(n,\mathbb{R})$ and D, as previously, is a diagonal matrix with positive elements. The two-polar decomposition is a by-product of the polar decomposition of $\mathrm{GL}^+(n,\mathbb{R})$,

$$\varphi = UA,$$

where $U \in \mathrm{SO}(n,\mathbb{R})$, thus, $U^T = U^{-1}$, and $A = A^T$ is a symmetric positively-definite matrix. It is well-known that such a decomposition is unique, whereas the two-polar one is charged with some multivaluedness. Green and Cauchy deformation tensors are then represented as follows:

$$G = \varphi^T\varphi = A^2, \qquad C = (\varphi^{-1})^T\varphi^{-1} = UA^{-2}U^{-1}.$$

One can also use the reversed polar decomposition

$$\varphi = BU, \qquad U \in \mathrm{SO}(n,\mathbb{R}), \qquad B = UAU^{-1} = B^T.$$

Then

$$G = U^{-1}B^2U, \qquad C = B^{-2}.$$

The two-polar decomposition is achieved by the orthogonal diagonalization of the matrix A,

$$A = VDV^{-1}, \qquad V \in \mathrm{SO}(n,\mathbb{R}).$$

Then

$$L = UV, \qquad R = V.$$

The polar splitting was described above in an over-simplified standard way, namely, U and V were identified with $\mathbb{R}^n$ and $\mathrm{LI}(U,V)$ with $\mathrm{GL}(n,\mathbb{R})$. Let us remind that in continuum mechanics the connected components of $\mathrm{LI}(U,V)$ and $\mathrm{GL}(n,\mathbb{R})$ are used as configuration spaces, $\mathrm{LI}^+(U,V)$ and $\mathrm{GL}^+(n,\mathbb{R})$, where the first symbol denotes the manifold of orientation-preserving isomorphisms (it is assumed here that some orientations in U, V are fixed). It is instructive to see what the both polar splittings are from the geometric point of view, when U and V are distinct linear spaces, non-identified with $\mathbb{R}^n$.

6.4 Geometric point of view on polar splittings

As mentioned above, when metric tensors $\eta \in U^* \otimes U^*$, $g \in V^* \otimes V^*$ are fixed, then any $\varphi \in \mathrm{LI}(U, V)$ with non-degenerate spectra of deformation tensors gives rise to the pair of orthonormal bases $(L_a[\varphi] \in V,\ a = \overline{1,n})$, $(R_a[\varphi] \in U,\ a = \overline{1,n})$. There exists exactly one isometry $U[\varphi] : U \to V$ such that

$$U[\varphi] \cdot R_a[\varphi] = L_a[\varphi].$$

Obviously, the isometry property is meant in the sense that $\eta = U[\varphi]^* \cdot g$, i.e., analytically

$$\eta_{AB} = g_{ij} U[\varphi]^i{}_A U[\varphi]^j{}_B.$$

Geometric meaning of the polar decomposition is as follows:

$$\varphi = U[\varphi]A[\varphi] = B[\varphi]U[\varphi],$$

where the automorphisms $A[\varphi] \in \mathrm{GL}(U)$, $B[\varphi] \in \mathrm{GL}(V)$ are symmetric, respectively, in the η- and g-sense, i.e.,

$$\eta(A[\varphi]x, y) = \eta(x, A[\varphi]y), \qquad g(B[\varphi]w, z) = g(w, B[\varphi]z)$$

for arbitrary $x, y \in U$, $w, z \in V$. They are also positively definite,

$$\eta(A[\varphi]x, x) > 0, \qquad g(B[\varphi]w, w) > 0$$

for arbitrary non-null $x \in U$, $w \in V$.

In spite of the non-uniqueness contained in $L[\varphi]$, $R[\varphi]$, the mappings $U[\varphi]$, $A[\varphi]$, $B[\varphi]$ are unique. And the symmetric parts are obtained from each other by the $U[\varphi]$-intertwining,

$$B[\varphi] = U[\varphi]A[\varphi]U[\varphi]^{-1}.$$

In mechanics of discrete affine systems we are free to admit orientation-reversing isometries U or symmetric mappings A, B not necessarily positively-definite.

6.5 Non-uniqueness of two-polar decomposition

The non-uniqueness of the two-polar decomposition mentioned above is important in certain computational and also principal problems, so some comments are necessary here. The problem is technically complicated, thus, only necessary facts will be quoted here, some of them formulated in a rather brief, rough way.

The subgroup of $\mathrm{O}(n,\mathbb{R})$ consisting of matrices which have exactly one non-vanishing entry in every row and column will be denoted by K. Obviously, K is finite and the mentioned entries are ± 1, reals with absolute value 1. The subgroup of proper K-rotations will denoted by $K^+ := K \cap \mathrm{SO}(n,\mathbb{R})$. Obviously, the orders (numbers of elements) of K, K^+ equal respectively $2n \cdot n!$ and $n \cdot n!$. Let $W \in K$ be a corresponding similarity transformation preserving the group of diagonal matrices,

$$\mathrm{Diag}(\mathbb{R}^n) \ni D \mapsto W^{-1}DW \in \mathrm{Diag}(\mathbb{R}^n),$$

and resulting in permutation of diagonal elements of $D = \mathrm{diag}(Q^1,\ldots,Q^n)$,

$$(Q^1,\ldots,Q^n) \mapsto (Q^{\pi_W(1},\ldots,Q^{n)}),$$

i.e.,

$$(q^1,\ldots,q^n) \mapsto (q^{\pi_W(1},\ldots,q^{n)}),$$

where $Q^a = \exp(q^a)$. Obviously, the mapping $K \ni W \mapsto \pi_W \in \mathrm{S}^{(n)}$ is a $2n:1$ epimorphism of K onto the permutation group $\mathrm{S}^{(n)}$. Its restriction to K^+ has an n-element kernel. The non-uniqueness of representation of $\varphi \in \mathrm{GL}^+(n,\mathbb{R})$ through elements of $\mathrm{SO}(n,\mathbb{R}) \times \mathbb{R}^n \times \mathrm{SO}(n,\mathbb{R})$ depends strongly on the degeneracy of spectra of deformation tensors. The multivaluedness is discrete, thus, simplest in the case of simple spectra.

6.6 Non-degenerate spectra case

Let $\mathrm{GL}^{+(n)}(n,\mathbb{R}) \subset \mathrm{GL}^+(n,\mathbb{R})$ be the subset of φ-s with non-degenerate spectra of C, G. The corresponding subset $M^{(n)}$ of $\mathrm{SO}(n,\mathbb{R}) \times \mathbb{R}^n \times \mathrm{SO}(n,\mathbb{R})$ consists of such triplets $(L; q^1,\ldots,q^n; R)$ that all q^i-s are pairwise distinct.

The group K^+ may be faithfully realized by the following transformation group $H^{(n)}$ of $M^{(n)}$:

$$(L; q^1, \ldots, q^n; R) \mapsto (LW; q^{\pi_W(1}, \ldots, q^{n)}; RW).$$

Obviously, this transformation does not affect $\varphi = LDR^{-1}$. Therefore, we have a diffeomorphism

$$\mathrm{GL}^{+(n)}(n, \mathbb{R}) \simeq M^{(n)}/H^{(n)}.$$

Non-degenerate spectrum is a generic one, nevertheless the coincidence case must be also taken into account because some new qualities appear then and they are relevant for qualitative analysis of classical phase portraits and for quantum conditions on admissible wave functions.

6.7 Degenerate case

Let $\mathrm{GL}^{+(k;p_1,\ldots,p_k)} \subset \mathrm{GL}^+(n, \mathbb{R})$ consist of φ-s for which deformation tensors have $k \leq n$ different principal values, every one of them with the corresponding multiplicity p_σ, $\sum_{\sigma=1}^{k} p_\sigma = n$. And similarly, let $M^{(k;p_1,\ldots,p_k)}$ be the set of such triplets $(L; q^1, \ldots, q^n; R) \in \mathrm{SO}(n, \mathbb{R}) \times \mathbb{R}^n \times \mathrm{SO}(n, \mathbb{R})$ that there are only k different q^i-s with the same conditions concerning multiplicity. And now let us consider the transformation group $H^{(k;p_1,\ldots,p_k)}$ acting on $M^{(k;p_1,\ldots,p_k)}$ as follows:

$$(L; q^1, \ldots, q^n; R) \mapsto (LW; q^{\pi_W(1}, \ldots, q^{n)}; RW),$$

where W runs over the subgroup of $\mathrm{SO}(n, \mathbb{R})$ generated by K^+ and the subgroup $H^{(k;p_1,\ldots,p_k)} \subset \mathrm{SO}(n, \mathbb{R})$ composed of k blocks $p_\sigma \times p_\sigma$, every one given by the corresponding $\mathrm{SO}(p_\sigma, \mathbb{R})$. Then we have that

$$\mathrm{GL}^{+(k;p_1,\ldots,p_k)} \simeq M^{(k;p_1,\ldots,p_k)}/H^{(k;p_1,\ldots,p_k)}.$$

When $k < n$, then at least one of multiplicities is non-trivial and the resulting group $H^{(k;p_1,\ldots,p_k)}$ is continuous. The resulting quotient is lower-dimensional because of this continuity of the divisor transformation group.

In the physical case $n = 3$, we have obviously only two possibilities of the non-trivial blocks, namely the total $\mathrm{SO}(3, \mathbb{R})$ and $\mathrm{SO}(2, \mathbb{R}) \times \mathrm{SO}(1, \mathbb{R})$ (respectively, all three q^i's equal or two of them); obviously $\mathrm{SO}(1, \mathbb{R}) = \{1\}$.

In the extreme case $k = 1$, D is proportional to the $n \times n$ identity matrix and it is only the total LR^{-1} that is well-defined; on the other hand L, R separately are meaningless.

6.8 Using two-polar splitting

It is very convenient and instructive to express our Hamiltonians, kinetic energies and configuration metrics in terms of the two-polar splitting. The previous statements concerning the phase pictures become then much more lucid. Let us introduce some auxiliary quantities:

$$M := -\hat{\rho} - \hat{\tau}, \qquad N := \hat{\rho} - \hat{\tau}. \tag{6.5}$$

One can easily show that the second-order Casimir invariant $C(2)$ occurring in the main terms of our affine-affine, affine-metrical and metrical-affine kinetic Hamiltonians has the following form:

$$C(2) = \sum_a p_a^2 + \frac{1}{16}\sum_{a,b}\frac{(M^a{}_b)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}} - \frac{1}{16}\sum_{a,b}\frac{(N^a{}_b)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}}. \tag{6.6}$$

Obviously, M and N are antisymmetric in the Kronecker-delta sense,

$$M^a{}_b = -M_b{}^a = -g_{bk}g^{al}M^k{}_l,$$

$$N^a{}_b = -N_b{}^a = -g_{bk}g^{al}N^k{}_l.$$

The first term in (6.6) may be suggestively decomposed into the "relative" and the "over-all" ("centre of mass") parts:

$$\frac{1}{2n}\sum_{a,b}(p_a - p_b)^2 + \frac{p^2}{n}.$$

Obviously, $C(1) = p$.

6.9 Boundedness of general solution

For geodetic systems and for more general systems with potentials V depending only on deformation invariants, spin $S = \rho$ and vorticity $V = \tau$ are constants of motion and may be used for extracting from equations of motion some information concerning the general solution. Unlike this the quantities $\hat{\rho}$, $\hat{\tau}$, thus, also M, N, fail to be constants of motion except the special case $n = 2$, when the rotation group is Abelian. However, on the

level of qualitative analysis, the expression (6.6) based on $\hat{\rho}$, $\hat{\tau}$ is more convenient because it does not involve L, R-variables, i.e., rotational degrees of freedom of deformation tensors. Therefore, our Poisson bracket relations imply that on the level of variables q^a, p_a, $M^a{}_b$, $N^a{}_b$ equations of motion based on (5.10) (equivalently (4.30)), (5.11) (equivalently (4.31)), and (5.9) with A replaced by $I + A$ are identical. In particular, for geodetic incompressible models and for compressible models with stabilized dilatations there exists an open family of bounded (vibrating) solutions and an open family of non-bounded (decaying) solutions. The reason is that it is so for (5.9) with A replaced by $I + A$, and the additional terms proportional to S^2 or V^2 do not influence anything because they have vanishing Poisson brackets with q^a, p_a, $M^a{}_b$, $N^a{}_b$ and only those variables occur in H. The only difference appears when the evolution of L- and R-variables is taken into account. However, the corresponding configuration spaces $\mathrm{F}(V, g)$, $\mathrm{F}(U, \eta)$ are compact (they are manifolds of orthonormal frames) and do not influence the boundedness of orbits.

6.10 Lattice structure of our models

6.10.1 Hyperbolic Sutherland-like lattices

Let us observe that after substituting (6.6), the first main term of (5.10) (equivalently (4.30)), (5.11) (equivalently (4.31)), and (5.9) with A replaced by $I + A$ acquires the characteristic lattice structure,

$$\mathcal{T}_{\mathrm{latt}} = \frac{1}{2\alpha}\sum_a p_a^2 + \frac{1}{32\alpha}\sum_{a,b}\frac{(M^a{}_b)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}} - \frac{1}{32\alpha}\sum_{a,b}\frac{(N^a{}_b)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}}.$$

This expression resembles structurally the hyperbolic Sutherland n-body system on the straight line. Positions of the fictitious material points are given by deformation invariants q^a. The "particles" have identical masses and are indistinguishable. Unlike the hyperbolic Sutherland system, the coupling amplitudes $M^a{}_b$, $N^a{}_b$ are non-equal and non-constant; rather they are dynamical variables on the equal footing with q^a, p_a. The negative N-contribution to $\mathcal{T}_{\mathrm{latt}}$ describes the attractive forces between lattice points, whereas the positive M-term corresponds to repulsion. Under the appropriate initial conditions we have stable bounded vibrations without any use of the potential energy term. Therefore, the non-definiteness of $\mathcal{T}_{\mathrm{latt}}$ is not

only non-embarrassing, but just desirable as a tool for describing "elastic" vibrations on the basis of purely geodetic models. Let us observe that the purely affine-affine part of (5.10), (5.11) (equivalently (4.30), (4.31)), i.e., (5.9) with A replaced by $I + A$ (composed of its first two Casimir terms), splits in the following suggestive way into the binary $\mathrm{SL}(n,\mathbb{R})$-part and dilatational contribution:

$$\begin{aligned}\mathcal{T}^{\mathrm{aff}}_{\mathrm{int}} &= \frac{1}{2\alpha}C(2)+\frac{1}{2\beta}C(1)^2=\frac{1}{4\alpha n}\sum_{a,b}(p_a-p_b)^2\\ &+ \frac{1}{32\alpha}\sum_{a,b}\frac{(M^a{}_b)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}}-\frac{1}{32\alpha}\sum_{a,b}\frac{(N^a{}_b)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}}+\frac{n\alpha+\beta}{2n\alpha\beta}p^2,\end{aligned}$$

or, in a more explicit form,

$$\begin{aligned}\mathcal{T}^{\mathrm{aff}}_{\mathrm{int}} &= \frac{1}{4(I+A)n}\sum_{a,b}(p_a-p_b)^2+\frac{1}{32(I+A)}\sum_{a,b}\frac{(M^a{}_b)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}}\\ &- \frac{1}{32(I+A)}\sum_{a,b}\frac{(N^a{}_b)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}}+\frac{1}{2n(I+A+nB)}p^2.\end{aligned}\tag{6.7}$$

Obviously, for (5.10) (equivalently (4.30)) and (5.11) (equivalently (4.31)) we have, respectively,

$$\mathcal{T}^{\mathrm{aff-metr}}_{\mathrm{int}} = \mathcal{T}^{\mathrm{aff}}_{\mathrm{int}}+\frac{I}{2(I^2-A^2)}\|V\|^2,\tag{6.8}$$

$$\mathcal{T}^{\mathrm{metr-aff}}_{\mathrm{int}} = \mathcal{T}^{\mathrm{aff}}_{\mathrm{int}}+\frac{I}{2(I^2-A^2)}\|S\|^2.\tag{6.9}$$

Copmparing this with (5.10), (5.11), we conclude that

$$\begin{aligned}C_{\mathrm{SL}(n)}(2) &= \mathrm{Tr}(\sigma^2)=\mathrm{Tr}(\hat{\sigma}^2)\\ &= \frac{1}{2n}\sum_{a,b}(p_a-p_b)^2+\frac{1}{16}\sum_{a,b}\frac{(M^a{}_b)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}}-\frac{1}{16}\sum_{a,b}\frac{(N^a{}_b)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}}.\end{aligned}$$

The above expression is very suggestive because it expresses the quantity $C_{\mathrm{SL}(n)}(2)$ and the corresponding contribution to $\mathcal{T}_{\mathrm{int}}$, i.e., the metric tensor on the manifold of incompressible motions, as the sum of $n(n-1)/2$ two-dimensional clusters, i.e., $\mathbb{R}^2$-coordinate planes in $\mathbb{R}^n$. Incompressibility is

expressed by the fact that the invariants q^a and their conjugate momenta p_a enter the above formula through the shape-describing differences $(q^a - q^b)$ (ratios Q^a/Q^b) and $p_a - p_b$. This expression may be very convenient when studying invariant geodetic models on the projective group $\mathrm{Pr}(n, \mathbb{R})$, i.e., when dealing with the mechanics of projectively-rigid bodies (bodies subject to such constraints that all geometric relationships of projective geometry are preserved, in particular, the material straight lines remain straight lines). The point is that $\mathrm{Pr}(n, \mathbb{R})$ may be identified in a standard way with $\mathrm{SL}(n+1, \mathbb{R})$.

6.10.2 Calogero-Moser-like lattices

For the d'Alembert model the two-polar splitting leads to the following kinetic Hamiltonian term:

$$\mathcal{T}_{\mathrm{int}} = \frac{1}{2I}\sum_a P_a^2 + \frac{1}{8I}\sum_{a,b}\frac{(M^a{}_b)^2}{(Q^a - Q^b)^2} + \frac{1}{8I}\sum_{a,b}\frac{(N^a{}_b)^2}{(Q^a + Q^b)^2}. \qquad (6.10)$$

It is purely repulsive on the level of Q-variables, thus, without any potential term it is non-realistic as a model of elastic vibrations. It is related to the Calogero-Moser lattices similarly as the previous models show some kinship with the hyperbolic Sutherland lattices [10, 51, 52, 70, 71, 72, 81, 83].

6.10.3 Usual Sutherland-like lattices

What concerns affine models, we can compactify deformation invariants q^a by taking them modulo 2π (n-dimensional torus), i.e., by putting formally $Q^a = \exp(iq^a)$. This is equivalent to replacing $\mathrm{GL}(n, \mathbb{R})$ by $\mathrm{U}(n)$, i.e., another and completely opposite real form of $\mathrm{GL}(n, \mathbb{C})$. The Lie algebra $\mathrm{U}(n)'$ consists of anti-Hermitian matrices, and the positively definite kinetic energy may be postulated in the following form:

$$T_{\mathrm{int}} = -\frac{A}{2}\mathrm{Tr}(\Omega^2) - \frac{B}{2}(\mathrm{Tr}\Omega)^2 = \frac{A}{2}\mathrm{Tr}(\Omega^+\Omega) + \frac{B}{2}\mathrm{Tr}(\Omega^+)\mathrm{Tr}(\Omega),$$

where

$$\Omega = \frac{d\varphi}{dt}\varphi^{-1}, \qquad A > 0, \quad B > 0.$$

Obviously, in this expression for T, Ω may be as well replaced by

$$\hat{\Omega} = \varphi^{-1}\frac{d\varphi}{dt}.$$

Using again the "two-polar" decomposition $\varphi = LDR^{-1}$, where $L, R \in \mathrm{SO}(n,\mathbb{R})$, $D = \mathrm{diag}(\ldots, \exp(iq^a), \ldots)$, one obtains for the geodetic Hamiltonian:

$$\begin{aligned} \mathcal{T}_{\mathrm{int}} &= \frac{1}{2A}\sum_a p_a^2 + \frac{1}{32A}\sum_{a,b}\frac{(M^a{}_b)^2}{\sin^2\frac{q^a-q^b}{2}} \\ &+ \frac{1}{32A}\sum_{a,b}\frac{(N^a{}_b)^2}{\cos^2\frac{q^a-q^b}{2}} - \frac{B}{2A(A+nB)}p^2. \end{aligned} \tag{6.11}$$

The first three terms, corresponding to the $C(2)$-Casimir, resemble the usual Sutherland lattice for q-particles with the same provisos as previously. Geodetic motion is bounded, because $\mathrm{U}(n)$ is compact. Just as previously, it may be convenient to use the splitting into $\mathrm{SU}(n)$- and $\mathrm{U}(1)$-terms,

$$\begin{aligned} \mathcal{T}_{\mathrm{int}} &= \frac{1}{4An}\sum_{a,b}(p_a - p_b)^2 + \frac{1}{32A}\sum_{a,b}\frac{(M^a{}_b)^2}{\sin^2\frac{q^a-q^b}{2}} \\ &+ \frac{1}{32A}\sum_{a,b}\frac{(N^a{}_b)^2}{\cos^2\frac{q^a-q^b}{2}} + \frac{1}{2n(A+nB)}p^2. \end{aligned}$$

And, in particular,

$$C_{\mathrm{SU}(n)}(2) = \frac{1}{2n}\sum_{a,b}(p_a - p_b)^2 + \frac{1}{16}\sum_{a,b}\frac{(M^a{}_b)^2}{\sin^2\frac{q^a-q^b}{2}} + \frac{1}{16}\sum_{a,b}\frac{(N^a{}_b)^2}{\cos^2\frac{q^a-q^b}{2}}.$$

The binary structures of $C_{SL(n,\mathbb{R})}(2)$ and $C_{SU(n)}(2)$ and their dependence on the variables q^a, p_a through their differences $q^a - q^b$, $p_a - p_b$ is geometrically interesting in itself. The splitting into $\mathrm{SL}(2,\mathbb{R})$- and $SU(2)$-clusters corresponding to all possible coordinate planes $\mathbb{R}^2$ in $\mathbb{R}^n$ may be also analytically helpful. However, some sophisticated mathematical techniques would be necessary then, like, e.g., the Dirac procedure for degenerate/constrained system. The point is that, in general, different clusters are not analytically independent. And any procedure based on some ordering

of variables destroys the explicit binary structure and makes the structure of $\mathcal{T}$ rather obscure.

It is interesting that the general solution of $C(2)$-based geodetic models contains as a particular subfamily the general solution of the mentioned Calogero-Moser and Sutherland models. It is obtained by putting $N^a{}_b = 0$, and all $M^a{}_b$ with $b \neq a$ equal to some fixed constant M.

6.11 Potential affine models

As mentioned, we are particularly interested in geodetic affine models. Nevertheless, it is instructive to admit a wider class of Hamiltonians:

$$H = \mathcal{T} + V(q^1, \ldots, q^n), \tag{6.12}$$

where $\mathcal{T}$ is any of the kinetic energy models described above, and the potential V depends on φ only through the deformation invariants q^a. This means that it is isotropic both in the physical and material space. The mentioned non-uniqueness of the two-polar decomposition implies that V as a function on $\mathbb{R}^n$ must be permutation-invariant to represent a well-defined function on the configuration space. When the extra potential, e.g., elastic one, is admitted, then also the "usual" model of $\mathcal{T}$ based on the d'Alembert principle may be sensibly used for describing bounded elastic vibrations. Therefore, from now on all the above models of $\mathcal{T}_{\text{int}}$ (6.7), (6.8), (6.9), (6.10) are admitted, although of course the "non-usual" affine models (6.7), (6.8), (6.9) are still particularly interesting for us.

As mentioned, the most convenient way of discussing and solving equations of motion is that based on Poisson brackets,

$$\frac{dF}{dt} = \{F, H\},$$

where F runs over some maximal system of (functionally) independent functions on the phase space. The most convenient and geometrically distinguished choice is $q^a, p_a, M^a{}_b, N^a{}_b, L, R$ or, more precisely, some coordinates on $\mathrm{SO}(n, \mathbb{R})$ parameterizing L and R. In d'Alembert models Q^a, P_a are more convenient the q^a, p_a.

An important point is that $q^a, p_a, M^a{}_b, N^a{}_b$ generate some Poisson subalgebra, because their Poisson brackets may be expressed by them alone without any use of L, R-variables. And Hamiltonians also depend only

on $q^a, p_a, M^a{}_b$, $N^a{}_b$, whereas L, R are non-holonomically cyclic variables. This enables one to perform a partial reduction of the problem. In fact, the following subsystem of equations is closed:

$$\begin{aligned}
\frac{dq^a}{dt} &= \{q^a, H\} = \frac{\partial H}{\partial p_a}, \\
\frac{dp_a}{dt} &= \{p_a, H\} = -\frac{\partial H}{\partial q^a}, \\
\frac{dM^a{}_b}{dt} &= \{M^a{}_b, H\} = \{M^a{}_b, M^c{}_d\}\frac{\partial H}{\partial M^c{}_d} + \{M^a{}_b, N^c{}_d\}\frac{\partial H}{\partial N^c{}_d}, \\
\frac{dN^a{}_b}{dt} &= \{N^a{}_b, H\} = \{N^a{}_b, M^c{}_d\}\frac{\partial H}{\partial M^c{}_d} + \{N^a{}_b, N^c{}_d\}\frac{\partial H}{\partial N^c{}_d}.
\end{aligned} \tag{6.13}$$

Obviously,

$$\{q^a, p_b\} = 1, \qquad \{q^a, M^c{}_d\} = \{p_a, M^c{}_d\} = \{q^a, N^c{}_d\} = \{p_a, N^c{}_d\} = 0.$$

Poisson brackets of M, N-quantities follow directly from those for $\hat{\rho}$, $\hat{\tau}$, and the latter ones correspond exactly to the structure constants of $\mathrm{SO}(n, \mathbb{R})$, thus,

$$\begin{aligned}
\{\hat{\rho}_{ab}, \hat{\rho}_{cd}\} &= \hat{\rho}_{ad}\delta_{cb} - \hat{\rho}_{cb}\delta_{ad} + \hat{\rho}_{db}\delta_{ac} - \hat{\rho}_{ac}\delta_{db}, \\
\{\hat{\tau}_{ab}, \hat{\tau}_{cd}\} &= \hat{\tau}_{ad}\delta_{cb} - \hat{\tau}_{cb}\delta_{ad} + \hat{\tau}_{db}\delta_{ac} - \hat{\tau}_{ac}\delta_{db}, \\
\{\hat{\rho}_{ab}, \hat{\tau}_{cd}\} &= 0,
\end{aligned}$$

where the raising and lowering of indices are meant in the Kronecker-delta sense. From these Poisson brackets we obtain the following ones:

$$\begin{aligned}
\{M_{ab}, M_{cd}\} = \{N_{ab}, N_{cd}\} &= M_{cb}\delta_{ad} - M_{ad}\delta_{cb} + M_{db}\delta_{ac} - M_{ac}\delta_{db}, \\
\{M_{ab}, N_{cd}\} &= N_{cb}\delta_{ad} - N_{ad}\delta_{cb} + N_{ac}\delta_{db} - N_{db}\delta_{ac}.
\end{aligned}$$

6.12 Reduction procedure

The subsystem (6.13) may be in principle autonomously solvable. When the time dependence of $\hat{\rho} = (N - M)/2$ and $\hat{\tau} = -(N + M)/2$ is known, then performing the inverse Legendre transformation we can obtain the time dependence of angular velocities $\hat{\chi}$, $\hat{\vartheta}$:

$$\hat{\chi}^a{}_b = \frac{\partial H}{\partial \hat{\rho}^b{}_a}, \qquad \hat{\vartheta}^a{}_b = \frac{\partial H}{\partial \hat{\tau}^b{}_a}$$

(some care must be taken when differentiating with respect to skew-symmetric matrices). And finally the evolution of L, R is given by the following time-dependent systems:

$$\frac{dL}{dt} = L\hat{\chi}, \qquad \frac{dR}{dt} = R\hat{\vartheta}.$$

There is some very important consequence of this reduction procedure, i.e., in doubly-isotropic models spin S, vorticity V, and their magnitudes $\|S\|$, $\|V\|$ are constants of motion. Moreover, $\|S\|$ and $\|V\|$ have vanishing Poisson brackets with all quantities q^a, p_a, $M^a{}_b$, $N^a{}_b$. Therefore, on the level of these variables, all Hamiltonian systems (6.12) with the same doubly isotropic potential V and with three affine models of the kinetic energy (6.7), (6.8), (6.9) are identical. In particular, the solutions for variables q^a, p_a, $M^a{}_b$, $N^a{}_b$ coincide with those for the affine-affine model (6.7). And this applies, in particular, to the geodetic model (when $V = 0$) and to the geodetic shear model with extra imposed dilatations stabilized by $V_{\rm dil}(q)$, where $q = (q^1 + \cdots + q^n)/n$. And then, as mentioned, the argument about one-parameter subgroups and their cosets decides about the existence of open subsets of bounded and non-bounded trajectories. The only difference between various $\mathcal{T}$-models appears only on the level of L, R-degrees of freedom. But the compactness of the corresponding configuration spaces $\mathrm{F}(V, g)$, $\mathrm{F}(U, \eta)$ implies that this part of motion does not influence the property of the total orbits in $Q = \mathrm{LI}(U, V)$ to be bounded or non-bounded. One should stress that for the affine-metrical and metrical-affine geodetic models (6.8), (6.9) only exceptional solutions are given by one-parameter subgroups and their cosets (relative equilibria). Nevertheless, extracting from all possible one-parameter subgroups and their cosets their $(q^a, p_a, M^a{}_b, N^a{}_b)$-content, we obtain true statements concerning all three geodetic models (6.7), (6.8), (6.9).

6.13 Natural potentials

Our affine geodetic models (6.7), (6.8), (6.9) have a nice binary structure with an additional degree of freedom related to the motion of the centre q of deformation invariants q^a, $a = \overline{1,n}$. In practical applications this term in $\mathcal{T}_{\rm int}$ should be stabilized by some extra introduced dilatational potential. If we perturb geodetic models by admitting more general doubly-isotropic

potentials, then it follows from the mentioned structure of $\mathcal{T}_{\text{int}}$ that the most natural and computationally effective potentials will be those somehow adapted to the above splitting into shear and dilatation parts, i.e.,

$$V(q^1,\dots,q^n) = V_{\text{dil}}(q) + \frac{1}{2}\sum_{i,j} {V_{\text{sh}}}^{ij}(|q^i - q^j|).$$

Here the additional shear part is not only binary but, just as it should be, it is depending only on the relative positions of deformation invariants $|q^i - q^j|$ on $\mathbb{R}$ or on the circle $\mathrm{U}(1)$ when the group $\mathrm{U}(n)$ is used. Obviously, the model of

$$V_{\text{sh}} = \frac{1}{2}\sum_{i,j} {V_{\text{sh}}}^{ij}(|q^i - q^j|)$$

will be computationally effective only when the structure of functions ${V_{\text{sh}}}^{ij}$ will have something to do with

$$\operatorname{sh}\left(\frac{q^i - q^j}{2}\right), \qquad \operatorname{ch}\left(\frac{q^i - q^j}{2}\right), \qquad \sin\left(\frac{q^i - q^j}{2}\right), \qquad \cos\left(\frac{q^i - q^j}{2}\right).$$

Chapter 7

Schroedinger quantization

A fascinating feature of our models of affine collective dynamics is their extremely wide range of applications. It covers the nuclear and molecular dynamics, micromechanics of structured continua, perhaps nanostructure and defects phenomena, macroscopic elasticity and astrophysical phenomena like vibration of stars and clouds of cosmic dust. Obviously, microphysical applications must be based on the quantized version of the theory. And one is dealing then with a very curious convolution of quantum theory with mathematical methods of continuum mechanics. It is worth to mention that there were even attempts, mainly by Barut and Rączka [6], to describe the dynamics of strongly interacting elementary particles (hadrons) in terms of some peculiar, quantized continua. By the way, as French say, the extremes teach one another; it is not excluded that the dynamics of cosmic objects like neutron stars must be also described in quantum terms. They are though giant nuclei, very exotic ones, because composed exclusively of neutrons (enormous "mass numbers" and vanishing "atomic numbers").

7.1 Quantization of classical geodetic systems

As usual, before quantizing the classical model, one has to perform some preliminary work on the level of its classical Hamiltonian dynamics [47, 73, 75, 76, 79].

Let us consider a classical geodetic system in a Riemannian manifold

(Q, Γ), where Q denotes the configuration space, and Γ is the "metric" tensor field on Q underlying the kinetic energy form. In terms of generalized coordinates we have

$$T = \frac{1}{2}\Gamma_{\mu\nu}\frac{dq^\mu}{dt}\frac{dq^\nu}{dt},$$

or in Hamiltonian terms

$$\mathcal{T} = \frac{1}{2}\Gamma^{\mu\nu}p_\mu p_\nu,$$

where, obviously,

$$\Gamma^{\mu\alpha}\Gamma_{\alpha\nu} = \delta^\mu{}_\nu$$

and

$$p_\mu = \frac{\partial T}{\partial \dot{q}^\mu} = \Gamma_{\mu\nu}\frac{dq^\nu}{dt}.$$

As usual, the metric tensor Γ gives rise to the natural measure μ_Γ on Q,

$$d\mu_\Gamma(q) = \sqrt{|\det[\Gamma_{\mu\nu}]|}dq^1 \cdots dq^f,$$

where f denotes the number of degrees of freedom, i.e., $f = \dim Q$. For simplicity the square-root expression will be always denoted by $\sqrt{|\Gamma|}$. The mathematical framework of Schrödinger quantization is based on $\mathrm{L}^2(Q, \mu_\Gamma)$, i.e., the Hilbert space of complex-valued wave functions on Q, which are square-integrable in the μ_Γ-sense. Their scalar product is given by the usual formula:

$$< \Psi_1|\Psi_2 >= \int \overline{\Psi}_1(q)\Psi_2(q)d\mu_\Gamma(q).$$

The classical kinetic energy expression is replaced by the operator

$$\mathbf{T} = -\frac{\hbar^2}{2}\Delta(\Gamma),$$

where $\hbar$ denotes the ("crossed") Planck constant, and $\Delta(\Gamma)$ is the Laplace-Beltrami operator corresponding to Γ,

$$\Delta(\Gamma) = \frac{1}{\sqrt{|\Gamma|}}\sum_{\mu,\nu}\partial_\mu\sqrt{|\Gamma|}\Gamma^{\mu\nu}\partial_\nu = \Gamma^{\mu\nu}\nabla_\mu\nabla_\nu.$$

In the last expression ∇_μ denotes the Levi-Civita covariant differentiation in the Γ-sense. Therefore, the kinetic energy operator $\mathbf{T}$ is formally obtained

from the corresponding classical expression $\mathcal{T}$ (kinetic Hamiltonian) by the substitution

$$p_\mu \mapsto \mathbf{p}_\mu = \frac{\hbar}{i}\nabla_\mu.$$

If the problem is non-geodetic and some potential $V(q)$ is admitted, the corresponding Hamilton (energy) operator is given by:

$$\mathbf{H} = \mathbf{T} + \mathbf{V},$$

where the operator $\mathbf{V}$ acts on wave functions simply multiplying them by V,

$$(\mathbf{V}\Psi)(q) = V(q)\Psi(q).$$

This is the reason why very often one does not distinguish graphically between $\mathbf{V}$ and V.

7.2 Problems concerning quantization

There are, obviously, many delicate problems concerning quantization which cannot be discussed here, and, fortunately, do not interfere directly with the main subjects of our analysis. Nevertheless, we mention briefly some of them. Strictly speaking, wave functions are not scalars but complex densities of the weight $1/2$ so that the bilinear expression $\overline{\Psi}\Psi$ is a real scalar density of weight one, thus, a proper object for describing probability distributions [39]. But in all realistic models, and the our one is not an exception, the configuration space is endowed with some Riemannian structure. And this enables one to factorize scalar (and tensor) densities into products of scalars (tensors) and some standard densities built of the metric tensor. Therefore, the wave function may be finally identified with the complex scalar field (multicomponent one when there are internal degrees of freedom).

There are also some arguments for modifying $\mathbf{T}$ by some scalar term proportional to the curvature scalar. Of course, such a term may be always formally interpreted as some correction potential. And besides, we usually deal with Riemannian manifolds of the constant Riemannian curvature, and then such additional terms result merely in the over-all shifting of energy levels.

In Riemann manifolds the Levi-Civita affine connection preserves the scalar product; because of this, the operator ∇_μ is formally anti-self-adjoint

and $(\hbar/i)\nabla_\mu$, $\mathbf{T} = -(\hbar^2/2)\Gamma^{\mu\nu}\nabla_\mu\nabla_\nu$ are formally self-adjoint. They are, however, differential operators, thus, the difficult problem of self-adjoint extensions appears. And besides, being differential operators, they are unbounded in the usual sense, thus, their spectral analysis also becomes a difficult and delicate subject. All such problems will be neglected and considered in the zeroth-order approximation of the mathematical rigor, just as it is usually done in practical physical applications. This is also justified by the fact that, as a rule, our first-order differential operators generate some well-definite global transformation groups admitting a lucid geometrical interpretation. It is typical that in such situation all subtle problems on the level of functional analysis, like the common domains, etc., may be successfully solved.

Therefore, from now on we will proceed in a "physical" way and all terms like "self-adjoint", "Hermitian", etc. will be used in a rough way characteristic for physical papers and applied mathematics.

7.3 Stationary situation

We shall deal almost exclusively with stationary problems when the Hamilton operator $\mathbf{H}$ is time-independent, thus, the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \mathbf{H}\psi$$

will be replaced by its stationary form, i.e., by the eigenequation

$$\mathbf{H}\Psi = E\Psi,$$

where, obviously,

$$\psi = \exp\left(-\frac{i}{\hbar}Et\right)\Psi$$

and Ψ is a time-independent wave function on the configuration space.

7.4 Multi-valuedness of wave functions

There is another delicate point concerning fundamental aspects of quantization which, however, may be of some importance and will be analyzed later on. Namely, it is claimed in all textbooks in quantum mechanics that

wave functions solving reasonable Schrödinger equations must satisfy strong regularity conditions, and first of all they must be well-defined one-valued functions all over the configuration space, in addition, continuous together with their derivatives. This demand is mathematically essential in the theory of Sturm-Liouville equations and besides it has to do with quantization or, more precisely, discrete spectra of certain physical quantities. By the way, these two things are not independent.

There are, however, certain arguments that some physical systems may admit multi-valued wave functions. It is so when the configuration space is not simply connected and its fundamental group is finite. Physically it is only the squared modulus $\overline{\Psi}\Psi$ that is to be one-valued because, according to the Born statistical interpretation, it represents the probability distribution of detecting a system in various regions of the configuration space. But for the wave function Ψ itself it is sufficient to be "locally" one-valued and sufficiently smooth, i.e., to be defined on the universal covering manifold $\overline{Q}$ of the configuration space Q. This may lead to a consistent quantum mechanics, perhaps with some kind of superselection rules. It is so in quantum mechanics of rigid body, which is sometimes expected to be a good model of the elementary particles spin [3, 4, 5]. The configuration space of the rigid body without translational motion may be identified with the proper rotation group $\mathrm{SO}(3,\mathbb{R})$ ($\mathrm{SO}(n,\mathbb{R})$ in n dimensions), obviously, when some reference orientation and Cartesian coordinates are fixed. But it is well-known that $\mathrm{SO}(3,\mathbb{R})$ is doubly-connected (and so is $\mathrm{SO}(n,\mathbb{R})$ for any $n \geq 3$). Its covering group is $\mathrm{SU}(2)$ ($\mathrm{Spin}(n)$ for any $n \geq 3$). Therefore, it is really an instructive exercise, and perhaps also a promising physical hypothesis, to develop the rigid top theory with $\mathrm{SU}(2)$ as configuration space [3, 4, 5]. In affinely-rigid body mechanics we are dealing with a similar situation, namely, $\mathrm{GL}(3,\mathbb{R})$ and $\mathrm{SL}(3,\mathbb{R})$ (more generally, $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{R})$ for $n > 3$) are doubly-connected. This topological property is simply inherited from the corresponding one for $\mathrm{SO}(3,\mathbb{R})$ ($\mathrm{SO}(n,\mathbb{R})$) on the basis of the polar decomposition [6, 90, 91]. Therefore, the standard quantization procedure in a manifold should be modified by using wave amplitudes defined on the covering manifolds $\overline{\mathrm{GL}(n,\mathbb{R})}$, $\overline{\mathrm{SL}(n,\mathbb{R})}$. By the way, some difficulty and mathematical curiosity appears then because these covering groups are non-linear (do not admit faithful realizations in terms of finite-dimensional matrices). This fact, known long ago to E. Cartan, was not known to physicists; a rather long time and enormous work has been lost because of this.

7.5 Classical background for quantization

Before going into such details we must go back to certain classical structures underlying quantization procedure. They were touched earlier in chapters 2 and 3 but in a rather superficial way, and besides, we concentrated there on the collective modes ruled by the linear and affine groups. This is really the main objective of our study, nevertheless, not exceptional one; it is also clear that, injecting the subject into a wider context, one attains a deeper understanding, free of accidental details.

In chapter 2 Lie-algebraic objects $\Omega, \hat{\Omega} \in G'$ were introduced. It is an important fact from the Lie group theory that they give rise to some vector fields X, Y on G invariant, respectively, under right and left translations on G. Namely, for any fixed $\Omega, \hat{\Omega} \in G'$, they are given by

$$X_g[\Omega] := \Omega g, \qquad Y_g[\hat{\Omega}] := g\hat{\Omega}.$$

Affine velocities introduced in chapter 3 are just the special case of Lie-algebraic objects. In the same chapter the dual objects Σ, $\hat{\Sigma}$, i.e., affine spin in two representations, were introduced. These dual quantities exist also in the general case when G is an arbitrary Lie group. They are then elements of the dual space, i.e., Lie co-algebra, $\Sigma, \hat{\Sigma} \in G'^*$. Their relationship with canonical momenta p and configurations g is given by the following formula involving evaluations of co-vectors on vectors:

$$< p, \dot{g} >=< \Sigma, \Omega >=< \hat{\Sigma}, \hat{\Omega} >,$$

where $\dot{g} \in T_gG$, $p \in T_g^*G$, and g, $\dot{g}$ are arbitrary. Denoting the adjoint transformation of Ad_g by the usual symbol Ad_g^*, we have that

$$\Sigma = \mathrm{Ad}_g^{*-1}\hat{\Sigma},$$

the obvious generalization of the corresponding relationship between laboratory and co-moving representation of affine (or usual metrical) spin. And just as in this special case, the quantities Σ, $\hat{\Sigma}$ are Hamiltonian generators of the groups of left and right regular translations L_G, R_G on G.

In applications we are usually dealing with some special Lie groups for which many important formulas and relationships may be written in a technically simple form avoiding the general abstract terms.

As mentioned, throughout this book we are dealing almost exclusively with linear groups $G \in \mathrm{GL}(W) \subset \mathrm{L}(W)$, where W is a linear space, e.g., some $\mathbb{R}^n$ or $\mathbb{C}^n$.

All the mentioned simplifications follow from the obvious canonical isomorphism between $\mathrm{L}(W)$ and its dual $\mathrm{L}(W)^*$, based on the pairing

$$< C, D >= \mathrm{Tr}\,(CD)\,.$$

The Lie algebra G' is a linear subspace of $\mathrm{L}(W)$, therefore, its dual space G'^* may be canonically identified with the quotient space $\mathrm{L}(W)^*/\mathrm{An}G'$, where $\mathrm{An}G'$ denotes the subspace of linear functions vanishing on G'. But, according to the above identification between $\mathrm{L}(W)^*$ and $\mathrm{L}(W)$ itself, $\mathrm{An}G'$ may be identified with some linear subspace of $\mathrm{L}(W)$; we shall denote it by $G'^{\perp}$. Therefore, the Lie co-algebra G'^* is canonically isomorphic with the corresponding quotient, i.e.,

$$G'^* \simeq \mathrm{L}(W)/G'^{\perp}.$$

This is the general fact for linear groups and their Lie algebras. However, in some special cases, just ones of physical relevance, this quotient space admits a natural canonical isomorphism onto some distinguished linear subspace of $\mathrm{L}(W)$ consisting of natural representants of cosets, e.g., in the most practical cases G'^* is canonically isomorphic with G' itself. For example, it is so for $\mathrm{SO}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{R})$, where the Lie algebras $\mathrm{SO}(n,\mathbb{R})'$, $\mathrm{SL}(n,\mathbb{R})'$ may be identified with the duals $\mathrm{SO}(n,\mathbb{R})'^*$, $\mathrm{SL}(n,\mathbb{R})'^*$. By the way, for certain reasons it is more convenient to use the pairing

$$< A, B >= -\frac{1}{2}\mathrm{Tr}(AB)$$

for the orthogonal group $\mathrm{SO}(n,\mathbb{R})$.

Just as in the special case of affine objects, transformation rules for Σ, $\hat{\Sigma}$ are analogous to those for Ω, $\hat{\Omega}$; we mean transformations under regular translations:

$$\begin{aligned} L_k \quad &: \quad \Sigma \mapsto \mathrm{Ad}_k^{*-1}\Sigma, \qquad \hat{\Sigma} \mapsto \hat{\Sigma}, \\ R_k \quad &: \quad \Sigma \mapsto \Sigma, \qquad\quad \hat{\Sigma} \mapsto \mathrm{Ad}_k^{*}\hat{\Sigma}. \end{aligned}$$

Using the identifications mentioned above (assuming that they work), we can write these rules in a form analogous to that for non-holonomic velocities,

$$\begin{aligned} L_k \quad &: \quad \Sigma \mapsto k\Sigma k^{-1}, \qquad \hat{\Sigma} \mapsto \hat{\Sigma}, \\ R_k \quad &: \quad \Sigma \mapsto \Sigma, \qquad \hat{\Sigma} \mapsto k^{-1}\hat{\Sigma}k, \end{aligned}$$

i.e., just as it is for the affine spin.

Geometrical meaning of Σ and $\hat{\Sigma}$ is that of the momentum mappings induced, respectively, by the group of left and right regular translations. And the relationship between two versions of Σ-objects is given as follows:

$$\Sigma = g\hat{\Sigma}g^{-1}.$$

The objects Σ and $\hat{\Sigma}$ may be also interpreted in terms of right- and left-invariant differential forms (co-vector fields), i.e., Maurer-Cartan forms A, B on the group G. Assuming the afore-mentioned identification, we can express A, B for any fixed Σ, $\hat{\Sigma}$ in the following forms:

$$A_g[\Sigma] = g^{-1}\Sigma, \qquad B_g[\hat{\Sigma}] = \hat{\Sigma}^{-1}g.$$

Just as in the special case of affine systems, Poisson bracket relations of Σ- and $\hat{\Sigma}$-components are given by structure constants of G. Those for $\hat{\Sigma}$ have opposite signs to those for Σ, and the mutual ones vanish (left regular translations commute with the right ones).

7.6 Hamiltonian systems on Lie group spaces

Geodetic Hamiltonian systems on Lie group spaces were studied by various research groups; let us mention, e.g., the prominent mathematicians like Hermann, Arnold, Mishchenko, Fomenko, and others. Obviously, the special stress was laid on models with kinetic energies (Riemann structures on G) invariant under left or right regular translations. As expected, models invariant simultaneously under left and right translations have some special properties and due to their high symmetries are computationally simplest.

From now on we assume that our configuration space Q is a Lie group G or, more precisely, its homogeneous space with trivial isotropy groups. Also in a more general situation when isotropy groups are nontrivial (even continuous) a large amount of analysis performed on group spaces remains useful.

Obviously, just as in the special case of affinely-rigid bodies, left- and right-invariant kinetic energies T are, respectively, quadratic forms of $\hat{\Omega}$ and Ω with constant coefficients. Their underlying Riemannian structures on G are locally flat if and only if G is Abelian.

In both theoretical and practical problems the Hamilton language based on Poisson brackets is much more lucid and efficient than that based on Lagrange equations. If besides of geodetic inertia the system is influenced only by potential forces derivable from some potential energy term $V(q)$, then, obviously, the classical Hamiltonian is given by the following expression:

$$H = \mathcal{T} + V(q) = \frac{1}{2}\Gamma^{\mu\nu}(q)p_\mu p_\nu + V(q).$$

It is very convenient to express the Hamiltonian and all other essential quantities in terms of non-holonomic velocities and their conjugate non-holonomic (Poisson-non-commuting) momenta.

Let $\{E_\mu\}$ be some basis in the Lie algebra G' and q^μ be the corresponding canonical coordinates of the first kind on G, i.e.,

$$g(q) = \exp\left(q^\mu E_\mu\right).$$

Lie-algebraic objects $\Omega, \hat{\Omega} \in G'$ will be, respectively, expanded as follows:

$$\Omega = \Omega^\mu E_\mu, \qquad \hat{\Omega} = \hat{\Omega}^\mu E_\mu.$$

Using the expansion coefficients Ω^μ, $\hat{\Omega}^\mu$ one obtains the following simple expressions for the left- and right-invariant kinetic energies:

$$T_{\text{left}} = \frac{1}{2}\mathcal{L}_{\mu\nu}\hat{\Omega}^\mu\hat{\Omega}^\nu, \qquad T_{\text{right}} = \frac{1}{2}\mathcal{R}_{\mu\nu}\Omega^\mu\Omega^\nu,$$

where the matrices $\mathcal{L}$, $\mathcal{R}$ are constant, symmetric, and non-singular. The positive definiteness problem is a more delicate matter, and there are some hyperbolic-signature structures of some relevance both for physics and pure geometry.

For potential systems Legendre transformation may be easily described with the use of non-holonomic objects, respectively,

$$\hat{\Sigma}_\mu = \frac{\partial T_{\text{left}}}{\partial \hat{\Omega}^\mu} = \mathcal{L}_{\mu\nu}\hat{\Omega}^\nu, \qquad \Sigma_\mu = \frac{\partial T_{\text{right}}}{\partial \Omega^\mu} = \mathcal{R}_{\mu\nu}\Omega^\nu,$$

where, obviously, $\hat{\Sigma}_\mu$, Σ_μ are expansion coefficients of $\hat{\Sigma}$, Σ with respect to the dual basis $\{E^\mu\}$ of the Lie co-algebra, i.e.,

$$\hat{\Sigma} = \hat{\Sigma}_\mu E^\mu, \qquad \Sigma = \Sigma_\mu E^\mu.$$

The resulting Hamiltonians have, respectively, the following forms:

$$H = \mathcal{T}_{\text{left}} + V(q) = \frac{1}{2}\mathcal{L}^{\mu\nu}\hat{\Sigma}_\mu\hat{\Sigma}_\nu + V(q),$$

$$H = \mathcal{T}_{\text{right}} + V(q) = \frac{1}{2}\mathcal{R}^{\mu\nu}\Sigma_\mu\Sigma_\nu + V(q),$$

where, obviously, the matrices $[\mathcal{L}^{\mu\nu}]$, $[\mathcal{R}^{\mu\nu}]$ are reciprocal to $[\mathcal{L}_{\mu\nu}]$, $[\mathcal{R}_{\mu\nu}]$.

If structure constants of G' with respect to the basis $\{E_\mu\}$ are defined according to the convention

$$[E_\mu, E_\nu] = E_\lambda C^\lambda{}_{\mu\nu},$$

then the Poisson brackets of Σ-objects are given as follows:

$$\{\Sigma_\mu, \Sigma_\nu\} = \Sigma_\lambda C^\lambda{}_{\mu\nu}, \qquad \{\hat{\Sigma}_\mu, \hat{\Sigma}_\nu\} = -\hat{\Sigma}_\lambda C^\lambda{}_{\mu\nu}, \qquad \{\Sigma_\mu, \hat{\Sigma}_\nu\} = 0.$$

7.7 Basic differential operators

Let us define basic differential operators generating left and right regular translations on G. We denote them respectively by $\mathbf{L}_\mu$ and $\mathbf{R}_\mu$. Their action on complex- or vector-valued functions F on G is defined as follows:

$$(\mathbf{L}_\mu F)(g) := \frac{\partial}{\partial q^\mu} F\left(k(q)g\right)\Big|_{q=0}, \qquad (\mathbf{R}_\mu F)(g) := \frac{\partial}{\partial q^\mu} F\left(gk(q)\right)\Big|_{q=0}. \tag{7.1}$$

Their Lie-bracket (commutator) relations differ from the above Poisson rules for Σ-quantities by signs:

$$[\mathbf{L}_\mu, \mathbf{L}_\nu] = -\mathbf{L}_\lambda C^\lambda{}_{\mu\nu}, \qquad [\mathbf{R}_\mu, \mathbf{R}_\nu] = \mathbf{R}_\lambda C^\lambda{}_{\mu\nu}, \qquad [\mathbf{L}_\mu, \mathbf{R}_\nu] = 0.$$

Poisson brackets between Σ-objects and functions F depending only on coordinates q (pull-backs of functions defined on the configuration space $Q = G$) are given by

$$\{\Sigma_\mu, F\} = -\mathbf{L}_\mu F, \qquad \{\hat{\Sigma}_\mu, F\} = -\mathbf{R}_\mu F.$$

The system of Poisson brackets quoted above is sufficient for calculating any other Poisson bracket with the help of well-known properties of this

operation. Thus, e.g., for any pair of functions A, B depending in general on all phase-space variables we have the following expression:

$$\{A, B\} = \Sigma_\lambda C^\lambda{}_{\mu\nu} \frac{\partial A}{\partial \Sigma_\mu} \frac{\partial B}{\partial \Sigma_\nu} - \frac{\partial A}{\partial \Sigma_\mu} \mathbf{L}_\mu B + \frac{\partial B}{\partial \Sigma_\mu} \mathbf{L}_\mu A,$$

and, when the phase space is parameterized in terms of quantities q^μ, $\hat{\Sigma}_\mu$, we have the similar expression:

$$\{A, B\} = \hat{\Sigma}_\lambda C^\lambda{}_{\mu\nu} \frac{\partial A}{\partial \hat{\Sigma}_\mu} \frac{\partial B}{\partial \hat{\Sigma}_\nu} - \frac{\partial A}{\partial \hat{\Sigma}_\mu} \mathbf{R}_\mu B + \frac{\partial B}{\partial \hat{\Sigma}_\mu} \mathbf{R}_\mu A.$$

Obviously, the finite regular translations may be expressed in terms of the following exponential formulas:

$$F\left(k(q)g\right) = \exp\left(q^\mu \mathbf{L}_\mu\right) F, \qquad F\left(gk(q)\right) = \exp(q^\mu \mathbf{R}_\mu) F, \tag{7.2}$$

with all known provisos concerning exponentiation of differential operators.

Non-holonomic velocities Ω, $\hat{\Omega}$ depend linearly on generalized velocities $\dot{q}$,

$$\Omega^\mu = \Omega^\mu{}_\nu(q)\dot{q}^\nu, \qquad \hat{\Omega}^\mu = \hat{\Omega}^\mu{}_\nu(q)\dot{q}^\nu.$$

Similarly, Σ and $\hat{\Sigma}$ depend contragradiently on the conjugate momenta p,

$$\Sigma_\mu = p_\alpha \Sigma^\alpha{}_\mu(q), \qquad \hat{\Sigma}_\mu = p_\alpha \hat{\Sigma}^\alpha{}_\mu(q),$$

where, obviously,

$$\Sigma^\alpha{}_\mu \Omega^\mu{}_\beta = \delta^\alpha{}_\beta, \qquad \hat{\Sigma}^\alpha{}_\mu \hat{\Omega}^\mu{}_\beta = \delta^\alpha{}_\beta.$$

This leads to the following expressions for generators:

$$\mathbf{L}_\mu = \Sigma^\alpha{}_\mu \frac{\partial}{\partial q^\alpha}, \qquad \mathbf{R}_\mu = \hat{\Sigma}^\alpha{}_\mu \frac{\partial}{\partial q^\alpha}.$$

7.8 Remarks

Many of the above statements remain true for the general non-holonomic velocities and their conjugate momenta without group-theoretical background [22]. Nevertheless, there are also important facts depending on the group structure and on the properties of Σ_μ, $\hat{\Sigma}_\mu$ respectively as the basic right-

and left-invariant co-vector fields (Maurer-Cartan forms). This concerns mainly invariant volumes, scalar products, Hermiticity of basic operators, and structure of the Laplace-Beltrami operator.

In group manifolds we are usually interested in left- or right-invariant kinetic energies. Even in the special case of the double invariance the definition-based direct calculation of the corresponding Laplace-Beltrami operator and the volume element may be rather complicated. However, if the corresponding kinetic metrics is left- or right-invariant, then so is the resulting volume element. Therefore, the L^2-structure on G may be directly based on the integration with respect to the Haar measure. As known from the theory of locally compact groups, this measure is unique up to the constant normalization factor. In the special case of compact groups this normalization may be fixed by the natural demand that the total (finite in this case) volume equals to unity. In any case, the normalization is non-essential. In applications one deals usually with so-called unimodular groups, where the left and right measures coincide [38, 50]. Obviously, for the left- or right-invariant kinetic energies the measures μ_Γ built of the underlying metrics Γ are also left- or right-invariant. Therefore, they coincide with the Haar measure. This enables one to use the Haar measure from the very beginning as the integration prescription underlying the scalar product definition. This is very convenient for two reasons. First of all, for typical Lie groups appearing in physical applications the Haar measures are explicitly known. Another nice and reasonable feature of such a procedure is that once fixing the normalization we are given the standard integration procedure, whereas the use of $d\mu_\Gamma = \sqrt{|\Gamma|}dq^1 \cdots dq^f$ changes the scalar product normalization for various models of T (of Γ). This constant factor change is not very essential, but its dependence on various inertial parameters like the above I, A, B obscures the comparison of various models.

7.9 Unitary transformations

It follows from the very nature of the Haar measure μ that on the level of wave functions the left and right regular translations are realized by unitary transformations on $\mathrm{L}^2(G, \mu)$. More precisely, let us define for any $k \in G$ the operators $\mathbf{L}(k)$, $\mathbf{R}(k)$ given by

$$(\mathbf{L}(k)\Psi)(g) := \Psi(kg), \qquad (\mathbf{R}(k)\Psi)(g) := \Psi(gk)$$

for any $g \in G$. It is clear that $\mathbf{L}(k)$, $\mathbf{R}(k)$ preserve the space $\mathrm{L}^2(G,\mu)$, moreover, they are unitary transformations,

$$\langle \mathbf{L}(k)\Psi_1|\mathbf{L}(k)\Psi_2\rangle = \langle \mathbf{R}(k)\Psi_1|\mathbf{R}(k)\Psi_2\rangle = \langle \Psi_1|\Psi_2\rangle\,.$$

The assignments $G \ni k \mapsto \mathbf{L}(k), \mathbf{R}(k)$ are, respectively, a unitary anti-representation and representation of G in $\mathrm{L}^2(G,\mu)$,

$$\mathbf{L}(k_1k_2) = \mathbf{L}(k_2)\mathbf{L}(k_1), \qquad \mathbf{R}(k_1k_2) = \mathbf{R}(k_1)\mathbf{R}(k_2).$$

To convert $\mathbf{L}$ into representation it is sufficient to replace $\Psi(kg)$ by $\Psi(k^{-1}g)$. Obviously, the difference is rather cosmetical and related to the conventions concerning the definition of the superposition of mappings. Nevertheless, any neglect may lead to the accumulation of sign errors and finally to numerically wrong results.

The operators $\mathbf{L}_\mu$, $\mathbf{R}_\mu$ generate the above representations, thus, we have

$$\mathbf{L}\left(\exp(q^\mu E_\mu)\right) = \exp(q^\mu \mathbf{L}_\mu), \qquad \mathbf{R}\left(\exp(q^\mu E_\mu)\right) = \exp(q^\mu \mathbf{R}_\mu),$$

with all known provisos concerning domains and exponents of evidently unbounded differential operators. It is important to remember that the left-hand sides are always well-defined bounded unitary operators acting on the whole $\mathrm{L}^2(G,\mu)$. Unlike this, $\mathbf{L}_\mu$, $\mathbf{R}_\mu$ act only on differentiable functions, they are unbounded, and the problems of domain and convergence appear on the right-hand sides of the above equations.

Unitarity of $\mathbf{L}$, $\mathbf{R}$ implies that their generators $\mathbf{L}_\mu$, $\mathbf{R}_\mu$ are formally anti-self-adjoint (physicists tell roughly: anti-Hermitian), i.e.,

$$\langle \mathbf{L}_\mu\Psi_1|\Psi_2\rangle = -\langle \Psi_1|\mathbf{L}_\mu\Psi_2\rangle\,, \qquad \langle \mathbf{R}_\mu\Psi_1|\Psi_2\rangle = -\langle \Psi_1|\mathbf{R}_\mu\Psi_2\rangle\,,$$

assuming that the left- and right-hand sides are well-defined (this is the case, e.g., for differentiable compactly supported functions on G).

Now, let us introduce the following operators:

$$\mathbf{\Sigma}_\mu := \frac{\hbar}{i}\mathbf{L}_\mu, \qquad \hat{\mathbf{\Sigma}}_\mu := \frac{\hbar}{i}\mathbf{R}_\mu. \tag{7.3}$$

They are formally self-adjoint, i.e., "Hermitian" in the rough language of quantum physicists:

$$\langle \mathbf{\Sigma}_\mu\Psi_1|\Psi_2\rangle = \langle \Psi_1|\mathbf{\Sigma}_\mu\Psi_2\rangle\,, \qquad \left\langle \hat{\mathbf{\Sigma}}_\mu\Psi_1|\Psi_2\right\rangle = \left\langle \Psi_1|\hat{\mathbf{\Sigma}}_\mu\Psi_2\right\rangle,$$

with the same as previously provisos concerning the functions Ψ_1, Ψ_2. Obviously, $\hbar$ denotes the ("crossed") Planck constant.

The operators $\mathbf{\Sigma}_\mu$, $\hat{\mathbf{\Sigma}}_\mu$ are quantized counterparts of classical physical quantities Σ_μ, $\hat{\Sigma}_\mu$. They may be expressed as follows:

$$\mathbf{\Sigma}_\mu = \frac{\hbar}{i}\Sigma^\alpha{}_\mu(q)\frac{\partial}{\partial q^\alpha}, \qquad \hat{\mathbf{\Sigma}}_\mu = \frac{\hbar}{i}\hat{\Sigma}^\alpha{}_\mu(q)\frac{\partial}{\partial q^\alpha}.$$

There is no problem of ordering of q-variables and differential operators $\partial/\partial q^\alpha$. This ordering is exactly as above, just due to the interpretation of $\mathbf{L}_\mu$ and $\mathbf{R}_\mu$ as infinitesimal generators of one-parameter subgroups.

7.10 Quantum Poisson bracket

In virtue of the above group-theoretical arguments the quantum Poisson-bracket rules are analogous to the classical ones,

$${}_Q\{\mathbf{\Sigma}_\mu, \mathbf{\Sigma}_\nu\} = \mathbf{\Sigma}_\lambda C^\lambda{}_{\mu\nu}, \qquad {}_Q\{\hat{\mathbf{\Sigma}}_\mu, \hat{\mathbf{\Sigma}}_\nu\} = -\hat{\mathbf{\Sigma}}_\lambda C^\lambda{}_{\mu\nu}, \qquad {}_Q\{\mathbf{\Sigma}_\mu, \hat{\mathbf{\Sigma}}_\nu\} = 0.$$

Let us remind that the quantum Poisson bracket of operators is defined as

$${}_Q\{\mathbf{A}, \mathbf{B}\} := \frac{1}{i\hbar}[\mathbf{A}, \mathbf{B}] = \frac{1}{i\hbar}(\mathbf{AB} - \mathbf{BA}).$$

One can show (see, e.g., [22]) that the kinetic energy operators for the left- and right-invariant models are given simply by the formerly quoted formulas with the classical generators Σ_μ, $\hat{\Sigma}_\mu$ replaced by the corresponding operators $\mathbf{\Sigma}_\mu$, $\hat{\mathbf{\Sigma}}_\mu$, i.e.,

$$\mathcal{T}_{\text{left}} = \frac{1}{2}\mathcal{R}^{\mu\nu}\hat{\mathbf{\Sigma}}_\mu\hat{\mathbf{\Sigma}}_\nu = -\frac{\hbar^2}{2}\mathcal{R}^{\mu\nu}\mathbf{R}_\mu\mathbf{R}_\nu,$$

$$\mathcal{T}_{\text{right}} = \frac{1}{2}\mathcal{L}^{\mu\nu}\mathbf{\Sigma}_\mu\mathbf{\Sigma}_\nu = -\frac{\hbar^2}{2}\mathcal{L}^{\mu\nu}\mathbf{L}_\mu\mathbf{L}_\nu.$$

As mentioned, the literal calculation of the Laplace-Beltrami operator in terms of local coordinates q^μ is usually very complicated and the resulting formula is, as a rule, quite obscure, non-readable, and because of this practically non-useful. Unlike this, the above block expression in terms of generators is geometrically lucid and well apt for solving procedure of the

Schrödinger equation. In various problems it is sufficient to operate algebraically with quantum Poisson brackets. To complete the above system of brackets let us quote expressions involving generators and position-type variables. The latter ones are operators which multiply wave functions by other functions on the configuration space,

$$(\mathbf{F}\Psi)(q) := F(q)\Psi(q).$$

If there is no danger of misunderstanding, we will not distinguish graphically between $\mathbf{F}$ and F. Just as on the classical level we have

$$_Q\{\mathbf{\Sigma}_\mu, \mathbf{F}\} = -\mathbf{L}_\mu F, \qquad _Q\{\hat{\mathbf{\Sigma}}_\mu, \mathbf{F}\} = -\mathbf{R}_\mu F.$$

Obviously, two position-type operators mutually commute.

Remark: Obviously, only for generators and position quantities the quantum and classical Poisson rules are identical. For other quantities it is no longer the case, moreover, there are problems with the very definition of quantum counterparts of other classical quantities. The very existence of the above distinguished family of physical quantities is due to the group-theoretical background of degrees of freedom.

7.11 Corresponding Haar measures

Let us now return to the main subject of our analysis, i.e., to the quantization of affine systems. For technical purposes we again fix some Cartesian coordinates x^i, a^K in M, N and identify analytically the configuration space $Q = \mathrm{LI}(U,V) \times M$ with the affine group $\mathrm{GAf}(n,\mathbb{R}) \simeq \mathrm{GL}(n,\mathbb{R}) \times_s \mathbb{R}^n$. Similarly, the internal configuration space $Q_{\mathrm{int}} = \mathrm{LI}(U,V)$ is identified with $\mathrm{GL}(n,\mathbb{R})$. The corresponding Haar measures will be denoted respectively by α, λ, i.e.,

$$\begin{aligned} d\alpha(\varphi, x) &= (\det\varphi)^{-n-1} dx^1 \cdots dx^n d\varphi^1{}_1 \cdots d\varphi^n{}_n \\ &= (\det\varphi)^{-1} d\lambda(\varphi) dx^1 \cdots dx^n, \\ d\lambda(\varphi) &= (\det\varphi)^{-n} d\varphi^1{}_1 \cdots d\varphi^n{}_n. \end{aligned}$$

In terms of the binary decomposition we have the following expression:

$$d\lambda(\varphi) = d\lambda(l;q;r) = \prod_{i\neq j} \left|\mathrm{sh}(q^i - q^j)\right| d\mu(l) d\mu(r) dq^1 \cdots dq^n,$$

where μ denotes the Haar measure on $\mathrm{SO}(n,\mathbb{R})$. Due to the compactness of $\mathrm{SO}(n,\mathbb{R})$ we can, but of course need not, normalize μ to unity, $\mu\left(\mathrm{SO}(n,\mathbb{R})\right)=1$.

The Haar measure on $\mathrm{SL}(n,\mathbb{R})$ used in quantum mechanics of incompressible objects may be symbolically written with the use of Dirac distribution as follows:

$$d\lambda_{SL}(\varphi)=\prod_{i\neq j}\left|\mathrm{sh}(q^i-q^j)\right|d\mu(l)d\mu(r)\delta(q^1+\cdots+q^n)dq^1\cdots dq^n.$$

7.12 Kinetic energy operators for affine models

Affine spin and its co-moving representation are, respectively, given by the following formally self-adjoint operators:

$$\mathbf{\Sigma}^a{}_b:=\frac{\hbar}{i}\mathbf{L}^a{}_b=\frac{\hbar}{i}\varphi^a{}_K\frac{\partial}{\partial\varphi^b{}_K},\qquad \hat{\mathbf{\Sigma}}^A{}_B:=\frac{\hbar}{i}\mathbf{R}^A{}_B=\frac{\hbar}{i}\varphi^m{}_B\frac{\partial}{\partial\varphi^m{}_A}.$$

The usual spin and vorticity operators are respectively given by

$$\mathbf{S}^a{}_b:=\mathbf{\Sigma}^a{}_b-g^{ac}g_{bd}\mathbf{\Sigma}^d{}_c,\qquad \mathbf{V}^A{}_B:=\hat{\mathbf{\Sigma}}^A{}_B-\eta^{AC}\eta_{BD}\hat{\mathbf{\Sigma}}^D{}_C. \tag{7.4}$$

Kinetic energy operators corresponding to the formerly described classical models of internal kinetic energies are simply obtained by replacing the classical quantities $\Sigma^a{}_b$, $\hat{\Sigma}^A{}_B$ by the above operators $\mathbf{\Sigma}^a{}_b$, $\hat{\mathbf{\Sigma}}^A{}_B$ without any attention to be paid to the ordering problem (just because of the group-theoretic interpretation of these quantities).

Thus, for the affine-affine model (affine both in space and in the material) we have

$$\begin{aligned}\mathbf{T}^{\mathrm{aff-aff}}_{\mathrm{int}} &= \frac{1}{2A}\mathbf{\Sigma}^i{}_j\mathbf{\Sigma}^j{}_i-\frac{B}{2A(A+nB)}\mathbf{\Sigma}^i{}_i\mathbf{\Sigma}^j{}_j\\ &= \frac{1}{2A}\hat{\mathbf{\Sigma}}^A{}_B\hat{\mathbf{\Sigma}}^B{}_A-\frac{B}{2A(A+nB)}\hat{\mathbf{\Sigma}}^A{}_A\hat{\mathbf{\Sigma}}^B{}_B.\end{aligned}$$

Similarly, for models with the mixed metrical-affine and affine-metrical in-

variance we have, respectively,

$$\begin{aligned}\mathbf{T}^{\mathrm{met-aff}}_{\mathrm{int}} &= \frac{1}{2\tilde{I}}g_{ik}g^{jl}\mathbf{\Sigma}^i{}_j\mathbf{\Sigma}^k{}_l + \frac{1}{2\tilde{A}}\mathbf{\Sigma}^i{}_j\mathbf{\Sigma}^j{}_i + \frac{1}{2\tilde{B}}\mathbf{\Sigma}^i{}_i\mathbf{\Sigma}^j{}_j,\\ \mathbf{T}^{\mathrm{aff-met}}_{\mathrm{int}} &= \frac{1}{2\tilde{I}}\eta_{AB}\eta^{CD}\hat{\mathbf{\Sigma}}^A{}_C\hat{\mathbf{\Sigma}}^B{}_D + \frac{1}{2\tilde{A}}\hat{\mathbf{\Sigma}}^A{}_B\hat{\mathbf{\Sigma}}^B{}_A + \frac{1}{2\tilde{B}}\hat{\mathbf{\Sigma}}^A{}_A\hat{\mathbf{\Sigma}}^B{}_B\end{aligned}$$

with the same as previously meaning of symbols $\tilde{I}$, $\tilde{A}$, $\tilde{B}$.

Similarly, the corresponding expressions for $\mathcal{T}_{\mathrm{tr}}$ have the following forms:

$$\begin{aligned}\mathbf{T}^{\mathrm{met-aff}}_{\mathrm{tr}} &= \frac{m}{2}g^{ij}\mathbf{P}_i\mathbf{P}_j = \frac{m}{2}\widetilde{G}^{AB}\hat{\mathbf{P}}_A\hat{\mathbf{P}}_B,\\ \mathbf{T}^{\mathrm{aff-met}}_{\mathrm{tr}} &= \frac{m}{2}\widetilde{C}^{ij}\mathbf{P}_i\mathbf{P}_j = \frac{m}{2}\eta^{AB}\hat{\mathbf{P}}_A\hat{\mathbf{P}}_B,\end{aligned}$$

where $\mathbf{P}_i$, $\hat{\mathbf{P}}_A$ are linear momentum operators respectively in laboratory and co-moving representations,

$$\mathbf{P}_a = \frac{\hbar}{i}\frac{\partial}{\partial x^a}, \qquad \hat{\mathbf{P}}_K = \varphi^a{}_K\mathbf{P}_a = \frac{\hbar}{i}\varphi^a{}_K\frac{\partial}{\partial x^a}.$$

Just as previously, $\widetilde{C}$, $\widetilde{G}$ are contravariant reciprocals of deformation tensors,

$$\widetilde{C}^{ik}C_{kj} = \delta^i{}_j, \qquad \widetilde{G}^{AC}G_{CB} = \delta^A{}_B.$$

As mentioned, there are no affine-affine models of $\mathbf{T}_{\mathrm{tr}}$, and therefore, no affine-affine models of $\mathbf{T}$. The corresponding "metric tensors" on $\mathrm{GAf}(n,\mathbb{R})$ would have to be singular.

Another important physical quantity is the canonical momentum conjugate to the dilatational coordinate q. On the quantum level it is represented by the formally self-adjoint operator

$$\mathbf{p} = \frac{\hbar}{i}\frac{\partial}{\partial q}.$$

It is also convenient to use the deviatoric (shear) parts of the affine spin,

$$\mathbf{s}^a{}_b := \mathbf{\Sigma}^a{}_b - \frac{\mathbf{p}}{n}\delta^a{}_b, \qquad \hat{\mathbf{s}}^A{}_B := \hat{\mathbf{\Sigma}}^A{}_B - \frac{\mathbf{p}}{n}\delta^A{}_B;$$

obviously,

$$\mathbf{p} = \mathbf{\Sigma}^a{}_a = \hat{\mathbf{\Sigma}}^A{}_A.$$

Due to the group-theoretical structure of the above objects as generators, the classical splitting of $\mathbf{T}$ into incompressible (shear-rotational) and dilatational parts remains literally valid, namely, we have the following expressions:

$$\begin{aligned}
\mathbf{T}^{\mathrm{aff-aff}}_{\mathrm{int}} &= \frac{1}{2A}\mathbf{C}_{\mathrm{SL}(n)}(2) + \frac{1}{2n(A+nB)}\mathbf{p}^2, \\
\mathbf{T}^{\mathrm{met-aff}}_{\mathrm{int}} &= \frac{1}{2(I+A)}\mathbf{C}_{\mathrm{SL}(n)}(2) \\
&+ \frac{1}{2n(I+A+nB)}\mathbf{p}^2 + \frac{I}{2(I^2-A^2)}\|\mathbf{S}\|^2, \\
\mathbf{T}^{\mathrm{aff-met}}_{\mathrm{int}} &= \frac{1}{2(I+A)}\mathbf{C}_{\mathrm{SL}(n)}(2) \\
&+ \frac{1}{2n(I+A+nB)}\mathbf{p}^2 + \frac{I}{2(I^2-A^2)}\|\mathbf{V}\|^2,
\end{aligned}$$

where, obviously,

$$\begin{aligned}
\mathbf{C}_{\mathrm{SL}(n)}(k) &:= \mathbf{s}^a{}_b\mathbf{s}^b{}_c\cdots\mathbf{s}^r{}_s\mathbf{s}^s{}_a \\
&= \hat{\mathbf{s}}^A{}_B\hat{\mathbf{s}}^B{}_C\cdots\hat{\mathbf{s}}^R{}_S\hat{\mathbf{s}}^S{}_A,
\end{aligned}$$

k terms in these expressions, and

$$\|\mathbf{S}\|^2 = -\frac{1}{2}\mathbf{S}^a{}_b\mathbf{S}^b{}_a, \qquad \|\mathbf{V}\|^2 = -\frac{1}{2}\mathbf{V}^A{}_B\mathbf{V}^B{}_A.$$

As mentioned, the $\mathrm{SL}(n,\mathbb{R})$-part of $\mathbf{T}$ has both discrete and continuous spectrum and predicts the bounded oscillatory solutions even if no extra potential on $\mathrm{SL}(n,\mathbb{R})$ is used (classically this is the geodetic model with an open subset of bounded trajectories in the complete solution). In particular, there is an open range of inertial parameters $(A,B,C)\in\mathbb{R}^3$ for which the spectrum is positive or at least bounded from below.

One can hope that on the basis of commutation relations for the Lie algebra $\mathrm{SL}(n,\mathbb{R})'$ some information concerning spectra and wave functions may be perhaps obtained without the explicit solving of differential equations.

There are $\mathrm{GL}(n,\mathbb{R})$-problems where the separation of isochoric $\mathrm{SL}(n,\mathbb{R})$-terms is not necessary, sometimes it is even undesirable. Then it is more

convenient to use the quantized version of (4.30), (4.31), (4.32), i.e.,

$$\begin{aligned}
\mathbf{T}^{\mathrm{aff}-\mathrm{aff}}_{\mathrm{int}} &= \frac{1}{2A}\mathbf{C}(2) - \frac{B}{2A(A+nB)}\mathbf{p}^2,\\
\mathbf{T}^{\mathrm{met}-\mathrm{aff}}_{\mathrm{int}} &= \frac{1}{2\alpha}\mathbf{C}(2) + \frac{1}{2\beta}\mathbf{p}^2 + \frac{1}{2\mu}\|\mathbf{S}\|^2,\\
\mathbf{T}^{\mathrm{aff}-\mathrm{met}}_{\mathrm{int}} &= \frac{1}{2\alpha}\mathbf{C}(2) + \frac{1}{2\beta}\mathbf{p}^2 + \frac{1}{2\mu}\|\mathbf{V}\|^2,
\end{aligned}$$

where α, β, μ are previously introduced constants in (4.26), (4.27), and $\mathbf{C}(k)$ are operators of the full $\mathrm{GL}(n,\mathbb{R})$-Casimirs,

$$\mathbf{C}(k) := \mathbf{\Sigma}^a{}_b\mathbf{\Sigma}^b{}_c\cdots\mathbf{\Sigma}^r{}_s\mathbf{\Sigma}^s{}_a = \hat{\mathbf{\Sigma}}^A{}_B\hat{\mathbf{\Sigma}}^B{}_C\cdots\hat{\mathbf{\Sigma}}^R{}_S\hat{\mathbf{\Sigma}}^S{}_A;$$

the above contracted products contain k terms. In particular,

$$\mathbf{C}(2) := \mathbf{\Sigma}^a{}_b\mathbf{\Sigma}^b{}_a = \hat{\mathbf{\Sigma}}^A{}_B\hat{\mathbf{\Sigma}}^B{}_A, \qquad \mathbf{C}(1) := \mathbf{\Sigma}^a{}_a = \hat{\mathbf{\Sigma}}^A{}_A.$$

In particular, if the inertial constant B vanishes, then the model $\mathbf{T}^{\mathrm{aff}-\mathrm{aff}}_{\mathrm{int}}$ may be interpreted in terms of one-dimensional multi-body problems in the sense of Calogero, Moser, Sutherland [56, 80], etc., quite independently of our primary motivation, i.e., n-dimensional affine systems.

As mentioned, on $\mathrm{GL}(n,\mathbb{R})$, i.e., for compressible objects with dilatations, some dilatation-stabilizing potential $V(q)$ must be introduced if the system has to possess bound states. For more general doubly isotropic potentials $V(q^1,\ldots,q^n)$ depending only on deformation invariants, there is no possibility of avoiding differential equations (with the help of ladder procedures). Nevertheless, the problem is then still remarkably simplified in comparison with the general case, because the quantum dynamics of deformation invariants is autonomous (in this respect the quantum problem is in a sense simpler than the classical one). The procedure is based then on the two-polar decomposition, which by the way is also very convenient on the level of purely geodetic models. In certain problems, e.g., spatially isotropic but materially anisotropic ones, the polar decomposition is also convenient.

7.13 Two-polar decomposition for quantum case

Let us go back to classical expressions (6.1), (6.2), (6.5). On the quantum level the classical quantities $\rho = S$, $\tau = -V$ become the operators of spin

and minus vorticity (7.4) $\mathbf{S}$, $-\mathbf{V}$, i.e., Hermitian generators of the unitary groups of spatial and material rotations (6.3) acting argument-wise on wave functions. Classical quantities $\hat{\rho}$, $\hat{\tau}$ were co-moving representants of tensors $\rho = S$, $\tau = -V$, i.e., their projections onto principal axes of the Cauchy and Green deformation tensors. Their quantum counterparts, i.e., operators $\hat{\mathbf{r}}$, $\hat{\mathbf{t}}$ are also co-moving representants of $\mathbf{r} = \mathbf{S}$, $\mathbf{t} = -\mathbf{V}$, i.e.,

$$\hat{\mathbf{r}}^a{}_b = L^a{}_i L^j{}_b \mathbf{S}^i{}_j, \qquad \hat{\mathbf{t}}^a{}_b = -R^A{}_b R^a{}_B \mathbf{V}^B{}_A. \tag{7.5}$$

They are Hermitian generators of the argument-wise right-hand side action (6.4) of $\mathrm{SO}(n,\mathbb{R})$ on the wave functions. Just as in classical theory, it is convenient to introduce operators

$$\mathbf{M}^a{}_b := -\hat{\mathbf{r}}^a{}_b - \hat{\mathbf{t}}^a{}_b, \qquad \mathbf{N}^a{}_b := \hat{\mathbf{r}}^a{}_b - \hat{\mathbf{t}}^a{}_b. \tag{7.6}$$

Commutation relations for operators $\mathbf{S}$, $\mathbf{V}$, $\hat{\mathbf{r}}$, $\hat{\mathbf{t}}$, $\mathbf{M}$, $\mathbf{N}$ are directly isomorphic with those for the generators of $\mathrm{SO}(n,\mathbb{R})$ and are expressed in a straightforward way in terms of $\mathrm{SO}(n,\mathbb{R})$-structure constants.

Now we are ready to write down explicitly our kinetic energy and Hamiltonian operators in terms of the two-polar splitting. We begin with the traditional integer spin models, and later on we show how half-integer angular momentum of extended bodies may appear in a natural way.

Quantum operators $\mathbf{S}^i{}_j$, $-\mathbf{V}^A{}_B$ have the following form:

$$\mathbf{S}^i{}_j = \frac{\hbar}{i}\mathbf{\Lambda}^i{}_j(L), \qquad -\mathbf{V}^A{}_B = \frac{\hbar}{i}\mathbf{\Lambda}^A{}_B(R), \tag{7.7}$$

where, according to the formulas (7.1), (7.2), (7.3), $\mathbf{\Lambda}^i{}_j(L)$ and $\mathbf{\Lambda}^A{}_B(R)$ are real first-order differential operators generating left regular translations on $\mathrm{SO}(n,\mathbb{R})$, or, more precisely, on the isometric factors $L : \mathbb{R}^n \to V$, $R : \mathbb{R}^n \to U$ of the two-polar splitting, i.e.,

$$\begin{aligned} F\left(W(\omega)L\right) &= \left(\exp\left(\frac{1}{2}\omega^j{}_i \mathbf{\Lambda}^i{}_j\right) F\right)(L), \\ F\left(W(\omega)R\right) &= \left(\exp\left(\frac{1}{2}\omega^B{}_A \mathbf{\Lambda}^A{}_B\right) F\right)(R). \end{aligned} \tag{7.8}$$

In the formulas above, F are functions on the manifolds of isometries from $(\mathbb{R}^n,\delta)$ to (V,g) and from $(\mathbb{R}^n,\delta)$ to (U,η). Analytically, in Cartesian coordinates they are simply functions on $\mathrm{SO}(n,\mathbb{R})$. Matrices $[\omega^a{}_b]$, $[\omega^A{}_B]$ are

respectively g- and η-antisymmetric:

$$\omega^a{}_b = -g^{ac}g_{bd}\omega^d{}_c, \qquad \omega^A{}_B = -\eta^{AC}\eta_{BD}\omega^D{}_C.$$

Their independent components are canonical coordinates of the first kind on $\mathrm{SO}(V,g)$, $\mathrm{SO}(U,\eta)$ (roughly, on $\mathrm{SO}(n,\mathbb{R})$),

$$W(\omega) = \exp\left(\frac{1}{2}\omega^b{}_a E^a{}_b\right), \qquad W(\omega) = \exp\left(\frac{1}{2}\omega^B{}_A E^A{}_B\right), \tag{7.9}$$

where $E^a{}_b \in \mathrm{SO}(V,g)'$, $E^A{}_B \in \mathrm{SO}(U,\eta)'$ are basic elements corresponding to some (arbitrary) choice of bases in V, U, i.e.,

$$(E^a{}_b)^i{}_j = \delta^a{}_j\delta^i{}_b - g^{ai}g_{bj}, \qquad (E^A{}_B)^C{}_D = \delta^A{}_D\delta^C{}_B - \eta^{AC}\eta_{BD}.$$

One could reproach against our permanent changing between the simplified analytical description based on $\mathbb{R}^n$, $\mathrm{GL}^+(n,\mathbb{R})$, $\mathrm{SO}(n,\mathbb{R})$ and the careful geometric distinguishing between the material and physical spaces U, V and the manifolds $\mathrm{LI}(U,V)$, $\mathrm{O}^+(\mathbb{R}^n,\delta;V,g)$, $\mathrm{O}^+(\mathbb{R}^n,\delta;U,\eta)$; the latter two denoting the manifolds of orientation-preserving isometries between indicated Euclidean spaces (equivalently, manifolds of positively oriented orthonormal frames $\mathrm{F}^+(V,g)$, $\mathrm{F}^+(U,\eta)$). However, this "monkey" way of changing branches has some advantages, provided that done carefully. There are relationships easily representable for computational purposes in matrix terms, however, in certain fundamental formulas this may be misleading and risky.

And now, at some final stage of our discussion there appear some expressions where the calculus on $\mathbb{R}^n$ as such (not on $\mathbb{R}^n$ base-identified with U, V) becomes not only temporarily admissible but just mathematically proper one. Namely, it is just the matrix group $\mathrm{SO}(n,\mathbb{R})$ that acts on the right on the objects $L \in \mathrm{O}^+(\mathbb{R}^n,\delta;V,g)$ and $R \in \mathrm{O}^+(\mathbb{R}^n,\delta;U,\eta)$. As said above, on the classical level the corresponding Hamiltonian generators, i.e., momentum mappings, are given by $[\hat{\rho}^a{}_b]$, $[\hat{\tau}^a{}_b]$. In quantized theory the same role is played by the formally self-adjoint differential operators $\hat{\mathbf{r}}^a{}_b$, $\hat{\mathbf{t}}^a{}_b$,

$$\begin{aligned} F\left(LW(\omega)\right) &= \left(\exp\left(\frac{1}{2}\omega^b{}_a\boldsymbol{\Upsilon}^a{}_b\right)F\right)(L) = \left(\exp\left(\frac{i}{2\hbar}\omega^b{}_a\hat{\mathbf{r}}^a{}_b\right)F\right)(L), \\ F\left(RW(\omega)\right) &= \left(\exp\left(\frac{1}{2}\omega^b{}_a\boldsymbol{\Upsilon}^a{}_b\right)F\right)(R) = \left(\exp\left(\frac{i}{2\hbar}\omega^b{}_a\hat{\mathbf{t}}^a{}_b\right)F\right)(R). \end{aligned} \tag{7.10}$$

Here the skew-symmetry of $[\omega^a{}_b]$ is meant in the literal Kronecker-delta sense; nothing like g and η is implicitly assumed:

$$\omega^a{}_b = -\omega_b{}^a = -\delta^{ac}\delta_{bd}\omega^d{}_c.$$

Just $\mathbb{R}^n$ as such with its numerical metric is used here. In the physical three-dimensional case one uses the duality between skew-symmetric tensors and axial vectors, thus, on the quantum operator level we use the quantities $\hat{\mathbf{r}}_a$, $\hat{\mathbf{t}}_a$, $\boldsymbol{\Upsilon}(L)_a$, $\boldsymbol{\Upsilon}(R)_a$, where

$$\hat{\mathbf{r}}^a{}_b = \epsilon^a{}_b{}^c\hat{\mathbf{r}}_c, \qquad \hat{\mathbf{r}}_a = \frac{1}{2}\epsilon_{ab}{}^c\hat{\mathbf{r}}^b{}_c,$$

$$\hat{\mathbf{t}}^a{}_b = \epsilon^a{}_b{}^c\hat{\mathbf{t}}_c, \qquad \hat{\mathbf{t}}_a = \frac{1}{2}\epsilon_{ab}{}^c\hat{\mathbf{t}}^b{}_c,$$

$$\boldsymbol{\Upsilon}^a{}_b = \epsilon^a{}_b{}^c\boldsymbol{\Upsilon}_c, \qquad \boldsymbol{\Upsilon}_a = \frac{1}{2}\epsilon_{ab}{}^c\boldsymbol{\Upsilon}^b{}_c.$$

Obviously, the expressions $\boldsymbol{\Upsilon}^a{}_b$, $\boldsymbol{\Upsilon}_a$ are meant in two versions, as acting on the L,R-variables, thus, puristically we should have used the symbols $\boldsymbol{\Upsilon}^a{}_b(L)$, $\boldsymbol{\Upsilon}_a(L)$, $\boldsymbol{\Upsilon}^a{}_b(R)$, $\boldsymbol{\Upsilon}_a(R)$, however, when non-necessary, we prefer to avoid the crowd of symbols. Commutation relations are in both cases:

$$[\boldsymbol{\Upsilon}_a, \boldsymbol{\Upsilon}_b] = \epsilon_{ab}{}^c\boldsymbol{\Upsilon}_c,$$

i.e., in terms of quantum Poisson brackets:

$$\frac{1}{i\hbar}[\hat{\mathbf{r}}_a, \hat{\mathbf{r}}_b] = -\epsilon_{ab}{}^c\hat{\mathbf{r}}_c, \qquad \frac{1}{i\hbar}[\hat{\mathbf{t}}_a, \hat{\mathbf{t}}_b] = -\epsilon_{ab}{}^c\hat{\mathbf{t}}_c.$$

It is clear that

$$[\hat{\mathbf{r}}_a, \hat{\mathbf{t}}_b] = 0, \qquad [\boldsymbol{\Upsilon}_a(L), \boldsymbol{\Upsilon}_b(R)] = 0.$$

Obviously, the raising and lowering of indices is meant here in the trivial Kronecker-delta sense, so it is written only for cosmetic reasons, e.g.,

$$\epsilon^a{}_b{}^c = \delta^{ak}\delta^{cl}\epsilon_{kbl},$$

etc. What concerns the V- and U-space objects like $\mathbf{S}^i{}_j = \mathbf{r}^i{}_j$, $\mathbf{V}^A{}_B = -\mathbf{t}^A{}_B$, analogous expressions are true when one uses orthonormal coordinates, i.e., when $g_{ij} =_* \delta_{ij}$, $\eta_{AB} =_* \delta_{AB}$. When more general rectilinear

coordinates are used, the formulas become more complicated because various expressions involving $\det[g_{ij}]$, $\det[\eta_{AB}]$ appear; there is, however, no practical need to use this representation.

In orthonormal coordinates in V and U spaces we have again the following expressions in terms of axial vectors:

$$\mathbf{r}^i{}_j = \mathbf{S}^i{}_j = \epsilon^i{}_j{}^k \mathbf{r}_k = \epsilon^i{}_j{}^k \mathbf{S}_k,$$

$$\mathbf{t}^A{}_B = -\mathbf{V}^A{}_B = \epsilon^A{}_B{}^C \mathbf{t}_C = -\epsilon^A{}_B{}^C \mathbf{V}_C.$$

These quantities are expressed through differential operators $\mathbf{\Lambda}^i{}_j(L)$ and $\mathbf{\Lambda}^A{}_B(R)$, cf. (7.7), for which the same dual representation will be used, i.e.,

$$\mathbf{\Lambda}^i{}_j(L) = \epsilon^i{}_j{}^k \mathbf{\Lambda}_k(L), \qquad \mathbf{\Lambda}_k(L) = \frac{1}{2}\epsilon_{ij}{}^k \mathbf{\Lambda}^j{}_k(L),$$

$$\mathbf{\Lambda}^A{}_B(R) = \epsilon^A{}_B{}^C \mathbf{\Lambda}_C(R), \qquad \mathbf{\Lambda}_A(R) = \frac{1}{2}\epsilon_{AB}{}^C \mathbf{\Lambda}^B{}_C(R).$$

When using the convention of "small" and "capital" indices, one can omit the L- and R-labels at $\mathbf{\Lambda}$-symbols. Obviously, we have:

$$\mathbf{S}_i = \mathbf{r}_i = \frac{\hbar}{i}\mathbf{\Lambda}_i, \qquad \mathbf{V}_A = -\mathbf{t}_A = -\frac{\hbar}{i}\mathbf{\Lambda}_A.$$

One should be careful with some subtle sign problems in commutation relations,

$$[\mathbf{\Lambda}_i, \mathbf{\Lambda}_j] = -\epsilon_{ij}{}^k \mathbf{\Lambda}_k, \qquad [\mathbf{\Lambda}_A, \mathbf{\Lambda}_B] = -\epsilon_{AB}{}^C \mathbf{\Lambda}_C, \qquad [\mathbf{\Lambda}_i, \mathbf{\Lambda}_A] = 0,$$

therefore,

$$\frac{1}{i\hbar}[\mathbf{S}_i, \mathbf{S}_j] = \epsilon_{ij}{}^k \mathbf{S}_k, \qquad \frac{1}{i\hbar}[\mathbf{V}_A, \mathbf{V}_B] = -\epsilon_{AB}{}^C \mathbf{V}_C, \qquad [\mathbf{S}_i, \mathbf{V}_A] = 0.$$

Let us also notice that

$$[\mathbf{\Lambda}_i, \mathbf{\Upsilon}_a(L)] = 0, \qquad [\mathbf{\Lambda}_i, \mathbf{\Lambda}_A] = 0, \qquad [\mathbf{\Lambda}_i, \mathbf{\Upsilon}_a(R)] = 0,$$

$$[\mathbf{\Lambda}_A, \mathbf{\Upsilon}_a(L)] = 0, \qquad [\mathbf{\Upsilon}_a(L), \mathbf{\Upsilon}_a(R)] = 0, \qquad [\mathbf{\Lambda}_A, \mathbf{\Upsilon}_a(R)] = 0.$$

7.14 "Rotation vector" space language

Obviously, "coordinates" $\omega^a{}_b$ on $\mathrm{SO}(n,\mathbb{R})$ are redundant, unless we restrict ourselves to $\omega_{ab} = \delta_{ac}\omega^c{}_b$, $a < b$ (or conversely). If $n = 3$, one uses so-called "rotation vector" k^a, where

$$\omega^a{}_b = -\epsilon^a{}_{bc}k^c, \qquad k^a = -\frac{1}{2}\epsilon^a{}_b{}^c\omega^b{}_c.$$

It is convenient to use the "magnitude" $k = \sqrt{(k^1)^2 + (k^2)^2 + (k^3)^2}$. In this parameterization, $\mathrm{SO}(3,\mathbb{R})$ is covered by the ball $k \leq \pi$ with the proviso that antipodal points on the sphere $k = \pi$ describe the same half-rotation, i.e., rotation by π about a given axis. For $k < \pi$ the representation is unique. The magnitude k equals the angle of rotation, whereas the versor $\overline{n} := \overline{k}/k$ represents the oriented rotation axis in the right screw sense (for $k = \pi$ it does not matter right or left ones; they coincide). In certain expressions it is convenient to use the spherical coordinates k, ϑ, φ in the $\overline{k}$-space, thus,

$$k^1 = k\sin\vartheta\cos\varphi, \qquad k^2 = k\sin\vartheta\sin\varphi, \qquad k^3 = k\cos\vartheta.$$

For the completeness, let us quote some important three-dimensional formulas.

The "basic" matrices $E^a{}_b \in \mathrm{SO}(3,\mathbb{R})'$ are represented dually by the actually basic system of E_a, where

$$E^a{}_b = \epsilon^a{}_b{}^c E_c, \qquad E_a = \frac{1}{2}\epsilon_{ab}{}^c E^b{}_c, \qquad (E_a)^b{}_c = -\epsilon_a{}^b{}_c.$$

The structure constants are then given simply by "epsilons":

$$[E_a, E_b] = \epsilon_{ab}{}^c E_c.$$

For any rotation vector $\overline{k} \in \mathbb{R}^3$, corresponding matrices $W\left(\overline{k}\right) \in \mathrm{SO}(3,\mathbb{R})$ act on vectors $\overline{u} \in \mathbb{R}^3$ as follows:

$$W\left(\overline{k}\right)\cdot\overline{u} = \cos k\overline{u} + \frac{(1-\cos k)}{k^2}\left(\overline{k}\cdot\overline{u}\right)\overline{k} + \frac{\sin k}{k}\overline{k}\times\overline{u};$$

obviously, the scalar and vector product are meant in the standard $\mathbb{R}^3$-sense. The components of $\overline{k}$ are canonical coordinates of the first kind on $\mathrm{SO}(3,\mathbb{R})$,

$$W\left(\overline{k}\right) = \exp\left(k^a E_a\right) = \sum_{m=0}^{\infty}\frac{1}{m!}\left(k^a E_a\right)^m.$$

One can show that

$$\begin{aligned} W\left(\overline{k}\right)\cdot\overline{u} &= \overline{u}+\overline{k}\times\overline{u}+\frac{1}{2}\overline{k}\times\left(\overline{k}\times\overline{u}\right)+\cdots \\ &+ \frac{1}{n!}\overline{k}\times\left(\overline{k}\times\left(\overline{k}\times\cdots\left(\overline{k}\times\overline{u}\right)\cdots\right)\right)+\cdots \end{aligned}$$

This infinite series is an alternative representation of the exponential formula. The term with multiplicator $1/n!$ contains the n-fold vector multiplication of $\overline{u}$ by $\overline{k}$. Explicitly the matrix of $W\left(\overline{k}\right)$ is given by

$$W\left(\overline{k}\right)^{a}{}_{b}=\cos k\ \delta^{a}{}_{b}+(1-\cos k)\frac{k^{a}k_{b}}{k^{2}}+\sin k\ \epsilon^{a}{}_{bc}\frac{k^{c}}{k};$$

obviously, the raising and lowering of indices is meant here in the trivial (purely cosmetic) delta-sense.

One can show that generators of right regular translations on $\mathrm{SO}(3,\mathbb{R})$ are given by the following expression:

$$\boldsymbol{\Upsilon}_{a}=\frac{k}{2}\mathrm{ctg}\frac{k}{2}\frac{\partial}{\partial k^{a}}+\left(1-\frac{k}{2}\mathrm{ctg}\frac{k}{2}\right)\frac{k_{a}k^{b}}{k^{2}}\frac{\partial}{\partial k^{b}}-\frac{1}{2}\epsilon_{ab}{}^{c}k^{b}\frac{\partial}{\partial k^{c}}.$$

This is a common formula for $\boldsymbol{\Upsilon}_{a}(L)$, $\boldsymbol{\Upsilon}_{a}(R)$, and now for simplicity we again use the analytical matrix representation, when U and V are identified with $\mathbb{R}^{3}$ and the L,R-terms of the two-polar decomposition are identified with elements of $\mathrm{SO}(3,\mathbb{R})$. To specify this formula to $\boldsymbol{\Upsilon}_{a}(L)$, $\boldsymbol{\Upsilon}_{a}(R)$ one must replace the general symbol of the rotation vector $\overline{k}$ on $\mathrm{SO}(3,\mathbb{R})$ by the rotation vectors $\overline{l}$, $\overline{r}$ parameterizing the L,R-terms:

$$L\left(\overline{l}\right)=\exp\left(l^{a}E_{a}\right),\qquad R\left(\overline{r}\right)=\exp\left(r^{a}E_{a}\right).$$

Generators of the left regular translations on $\mathrm{SO}(3,\mathbb{R})$ are as follows:

$$\boldsymbol{\Lambda}_{a}=\frac{k}{2}\mathrm{ctg}\frac{k}{2}\frac{\partial}{\partial k^{a}}+\left(1-\frac{k}{2}\mathrm{ctg}\frac{k}{2}\right)\frac{k_{a}k^{b}}{k^{2}}\frac{\partial}{\partial k^{b}}+\frac{1}{2}\epsilon_{ab}{}^{c}k^{b}\frac{\partial}{\partial k^{c}}.$$

And this again specifies to $\boldsymbol{\Lambda}_{a}(L)$, $\boldsymbol{\Lambda}_{a}(R)$ when instead of $\overline{k}$ we substitute respectively $\overline{l}$, $\overline{r}$, i.e., rotation vectors parameterizing the manifolds of L,R-factors in the two-polar decomposition.

Let us observe that

$$\boldsymbol{\Lambda}_{a}-\boldsymbol{\Upsilon}_{a}=\mathbf{D}_{a}=\epsilon_{ab}{}^{c}k^{b}\frac{\partial}{\partial k^{c}},$$

and these differential operators generate the group of inner automorphisms of $\mathrm{SO}(3,\mathbb{R})$:

$$W\left(\overline{k}\right) \mapsto UW\left(\overline{k}\right)U^{-1} = W\left(U\overline{k}\right),$$

where U runs over $\mathrm{SO}(3,\mathbb{R})$. Roughly speaking, these transformations result in rotations of the rotation vectors. And, just as previously, substituting here $\overline{l}$ and $\overline{r}$ in place of $\overline{k}$ we obtain the corresponding transformations of the manifolds of $L\left(\overline{l}\right)$- and $R\left(\overline{r}\right)$-terms of the two-polar decompositions. One can show that the generators of the left and right regular translations on $\mathrm{SO}(3,\mathbb{R})$ may be expressed in terms of operators $\partial/\partial k$ and $\mathbf{D}_a$ acting, respectively, along the radius and tangently to spheres in the representative spaces $\mathbb{R}^3$ of the rotation vector $\overline{k}$, i.e.,

$$\begin{aligned}\mathbf{\Lambda}_a &= \frac{k_a}{k}\frac{\partial}{\partial k} - \frac{1}{2}\mathrm{ctg}\frac{k}{2}\epsilon_{ab}{}^c k^b \mathbf{D}_c + \frac{1}{2}\mathbf{D}_a,\\ \mathbf{\Upsilon}_a &= \frac{k_a}{k}\frac{\partial}{\partial k} - \frac{1}{2}\mathrm{ctg}\frac{k}{2}\epsilon_{ab}{}^c k^b \mathbf{D}_c - \frac{1}{2}\mathbf{D}_a.\end{aligned}$$

Obviously,

$$[\mathbf{D}_a, \mathbf{D}_b] = -\epsilon_{ab}{}^c \mathbf{D}_c.$$

In many formulas we need orthogonal invariants like $\|\mathbf{S}\|^2$, $\|\mathbf{V}\|^2$. They are based on the Casimir invariants $C_{\mathrm{SO}(n,\mathbb{R})}(2)$ built of generators $\mathbf{\Lambda}_a$, $\mathbf{\Upsilon}_a$ of the left and right regular translations on $\mathrm{SO}(n,\mathbb{R})$. If $n = 3$, these Casimirs have the following form:

$$\mathbf{\Lambda}^2 = \mathbf{\Upsilon}^2 = \mathbf{\Lambda}_1^2 + \mathbf{\Lambda}_2^2 + \mathbf{\Lambda}_3^2 = \mathbf{\Upsilon}_1^2 + \mathbf{\Upsilon}_2^2 + \mathbf{\Upsilon}_3^2, \tag{7.11}$$

and one can show that analytically

$$\mathbf{C}_{\mathrm{SO}(3,\mathbb{R})}(2) = \mathbf{\Lambda}^2 = \mathbf{\Upsilon}^2 = \left(\frac{\partial^2}{\partial k^2} + \mathrm{ctg}\frac{k}{2}\frac{\partial}{\partial k}\right) + \frac{1}{4\sin^2\frac{k}{2}}\mathbf{D}^2,$$

where

$$\mathbf{D}^2 = \mathbf{D}_1^2 + \mathbf{D}_2^2 + \mathbf{D}_3^2.$$

Obviously,

$$\|\mathbf{S}\|^2 = -\hbar^2 \mathbf{C}_{\mathrm{SO}(3,\mathbb{R})}\left(L\left(\overline{l}\right)\right), \qquad \|\mathbf{V}\|^2 = -\hbar^2 \mathbf{C}_{\mathrm{SO}(3,\mathbb{R})}\left(R\left(\overline{r}\right)\right),$$

where the last two terms multiplied by $-\hbar^2$ are obtained from the previous $\mathbf{C}_{\mathrm{SO}(3,\mathbb{R})}$ by substituting the $\overline{l}$- and $\overline{r}$-variables in place of $\overline{k}$.

Remark: Obviously, the equality (7.11) of $\mathbf{\Lambda}^2$ and $\mathbf{\Upsilon}^2$ holds only when $\mathbf{\Lambda}_a$ and $\mathbf{\Upsilon}_a$ involve the same kind of independent variables, e.g., $\overline{k}$ on the abstract $\mathrm{SO}(3,\mathbb{R})$ as generators of the left or right regular translations, $\bar{l}$ when both operating on the left two-polar factor $L(\bar{l})$, or $\overline{r}$ when both acting on the right two-polar factor. But of course $\|\mathbf{S}\|^2$ and $\|\mathbf{V}\|^2$ are different for any dimension n, although, of course, the following always holds:

$$\|\mathbf{S}\|^2 = \|\hat{\mathbf{r}}\|^2, \qquad \|\mathbf{V}\|^2 = \|\hat{\mathbf{t}}\|^2,$$

just on the basis of equations (7.5).

7.15 Expansion of wave functions

When we use the two-polar decomposition $\varphi = LDR^{-1}$, then, according to the Peter-Weyl theorem, the wave functions on $\mathrm{GL}^+(n,\mathbb{R})$ may be expanded in L,R-variables with respect to matrix elements of irreducible representations of the compact group $\mathrm{SO}(n,\mathbb{R})$. Obviously, the expansion coefficients depend on deformation invariants, i.e., on the diagonal factor D (equivalently, on the variables Q^a or $q^a = \ln Q^a$). In general, we have that

$$\Psi(\varphi) = \Psi(L,D,R) = \sum_{\alpha,\beta\in\Omega} \sum_{m,n=1}^{N(\alpha)} \sum_{k,l=1}^{N(\beta)} \mathcal{D}^{\alpha}_{mn}(L) f^{\alpha\beta}_{\substack{nk\\ml}}(D) \mathcal{D}^{\beta}_{kl}(R^{-1}), \quad (7.12)$$

where the meaning of symbols is as follows: Ω is the set of equivalence classes of unitary irreducible representations of $\mathrm{SO}(n,\mathbb{R})$, $N(\alpha)$ is the dimension of the α-th representation class. It is finite because $\mathrm{SO}(n,\mathbb{R})$ is compact, $\mathcal{D}^{\alpha}$ is the α-th representation matrix. For many classical groups $\mathcal{D}^{\alpha}$ are explicitly known (at least in terms of some well-investigated special functions).

Analytically $\mathcal{D}^{\alpha}(L)$, $\mathcal{D}^{\beta}(R^{-1})$ are matrices depending on the group coordinates ${\omega_L}^a{}_b$, ${\omega_R}^a{}_b$ of L, R, e.g., rotation vectors $\bar{l}$, $\overline{r}$ if $n=3$. The argument D of f is the system of q-variables $q^1,\ldots,q^n$. According to the mentioned multi-valuedness of the two-polar decomposition, the reduced amplitudes $f^{\alpha\beta}(q^1,\ldots,q^n)$ must obey some conditions, because Ψ must not distinguish triplets (L,D,R) corresponding to the same configuration $\varphi = LDR^{-1}$.

Therefore, on the submanifold $M^{(n)} \subset \mathrm{SO}(n,\mathbb{R}) \times \mathbb{R}^n \times \mathrm{SO}(n,\mathbb{R})$ with non-degenerate systems of $(q^1,\ldots,q^n)$ (no coincidences) we must have that

$$f^{\alpha\beta}_{\substack{nk\\ml}}(q^{\pi_W(1)},\ldots,q^{n)}) = \sum_{r=1}^{N(\alpha)} \sum_{s=1}^{N(\beta)} \mathcal{D}^{\alpha}_{nr}(W^{-1}) f^{\alpha\beta}_{\substack{rs\\ml}}(q^1,\ldots,q^n) \mathcal{D}^{\beta}_{sk}(W)$$

for any $W \in K^+$. The same holds on the subsets $M^{(k;p_1,\dots,p_k)} \subset \mathrm{SO}(n,\mathbb{R}) \times \mathbb{R}^n \times \mathrm{SO}(n,\mathbb{R})$ with degenerate systems $(q^1,\dots,q^n)$ (coincidences of some q's). The difference is that in degenerate cases W runs over the continuous subgroups of $\mathrm{SO}(n,\mathbb{R})$ generated by K^+ and the subgroups $H^{(k;p_1,\dots,p_k)}$ described above. The special case of the total degeneracy is extreme and, because of this, very simple one. Indeed, then in the two-polar decomposition it is only LR^{-1} that is meaningful whereas L, R separately are not well-defined. Therefore, if $D = cI_n$, i.e., $q^1 = \dots = q^n = q$, then the reduced amplitude obeys very severe restrictions, i.e.,

$$\begin{aligned} f^{\alpha\beta}(cI_n) &= 0 \quad \text{if} \quad \alpha \neq \beta, \\ f^{\alpha\alpha}_{\substack{rs\\ml}}(cI_n) &= g_{ml}\delta_{rs}. \end{aligned}$$

The non-uniqueness is extreme here, namely, for any $Z \in \mathrm{SO}(n,\mathbb{R})$ the triplets (L, cI_n, R), (LZ, cI_n, RZ) represent the same classical configuration, thus, the wave functions do not distinguish them.

It is seen that if $q^1,\dots,q^n$ are interpreted as coordinates of some fictitious material points on the real axis $\mathbb{R}$, one is dealing with a very peculiar system of identical para-statistical particles.

It is clear that in geodetic models or in models with doubly isotropic potentials (ones depending only on deformation invariants; dilatation-stabilizing potentials $V(q)$ provide the simplest example), m and l in the Peter-Weyl expansion (7.12) are "good" quantum numbers. In other words, the spin and vorticity operators $\mathbf{S}^i{}_j$, $\mathbf{V}^A{}_B$ do commute with the Hamilton operator $\mathbf{H}$. The same concerns representation labels $\alpha, \beta \in \Omega$, i.e., finally, the systems of eigenvalues for the Casimir operators of the groups $\mathrm{SO}(V,g)$, $\mathrm{SO}(U,\eta)$ acting argument-wise on wave functions. Let us remind that these Casimirs are given by

$$\begin{aligned} \mathbf{C}_{\mathrm{SO}(V,g)}(p) &\simeq \mathbf{S}^i{}_k\mathbf{S}^k{}_m\cdots\mathbf{S}^r{}_z\mathbf{S}^z{}_i, \\ \mathbf{C}_{\mathrm{SO}(U,\eta)}(p) &\simeq \mathbf{V}^A{}_K\mathbf{V}^K{}_M\cdots\mathbf{V}^R{}_Z\mathbf{V}^Z{}_A, \end{aligned} \tag{7.13}$$

p operator multipliers in every expression; $p \leq n$ and even.

In such situation it is convenient to keep α, β, m, l fixed and use the following reduced amplitudes:

$$\Psi(\varphi) = \Psi^{\alpha\beta}_{ml}(L,D,R) = \sum_{n=1}^{N(\alpha)}\sum_{k=1}^{N(\beta)} \mathcal{D}^{\alpha}_{mn}(L) f^{\alpha\beta}_{nk}(D) \mathcal{D}^{\beta}_{kl}(R^{-1}), \tag{7.14}$$

with the same as previously provisos concerning the one-valuedness of Ψ as a function of φ.

In the physical case $n = 3$, we have obviously the following standard form of $\mathrm{SO}(3,\mathbb{R})$-Casimirs:

$$\mathbf{C}_{\mathrm{SO}(V,g)}(2) = \mathbf{S}_1^2 + \mathbf{S}_2^2 + \mathbf{S}_3^2 = \hat{\mathbf{r}}_1^2 + \hat{\mathbf{r}}_2^2 + \hat{\mathbf{r}}_3^2 = \mathbf{C}_{\mathrm{SO}(3,\mathbb{R})}(2),$$

$$\mathbf{C}_{\mathrm{SO}(U,\eta)}(2) = \mathbf{V}_1^2 + \mathbf{V}_2^2 + \mathbf{V}_3^2 = \hat{\mathbf{t}}_1^2 + \hat{\mathbf{t}}_2^2 + \hat{\mathbf{t}}_3^2 = \mathbf{C}_{\mathrm{SO}(3,\mathbb{R})}(2).$$

Our expansions for wave functions are then described in terms of well-known expressions found by Wigner, and, of course, the family of rotational Casimirs begins and terminates on $p = 2$.

Obviously, for $n = 3$, Ω is the set of non-negative integer, α, β are traditionally denoted by symbols like $s, j = 0, 1, 2, \ldots$, etc., $N(s) = 2s + 1$, $N(j) = 2j + 1$, and the indices (m, n), (k, l) are considered as jumping by 1, respectively, from $-s$ to s and from $-j$ to j; here the tradition is too strong to respect the formal logical conventions. Thus, the expansion (7.12) is written according to the mentioned conventions:

$$\Psi(\varphi) = \Psi(L, D, R) = \sum_{s,j=0}^{\infty} \sum_{m,n=-s}^{s} \sum_{k,l=-j}^{j} \mathcal{D}^s_{mn}(L) f^{sj}_{\substack{nk \\ ml}}(D) \mathcal{D}^j_{kl}(R^{-1}). \quad (7.15)$$

Similarly, the reduced amplitudes (7.14) are written as:

$$\Psi(\varphi) = \Psi^{sj}_{ml}(L, D, R) = \sum_{n=-s}^{s} \sum_{k=-j}^{j} \mathcal{D}^s_{mn}(L) f^{sj}_{nk}(D) \mathcal{D}^j_{kl}(R^{-1}). \quad (7.16)$$

Here $\mathcal{D}^s$ are celebrated Wigner matrices of $(2s+1)$-dimensional irreducible representations of the three-dimensional rotation group. They are well-known special functions of mathematical physics and may be assumed to be something in principle standard and well-know.

Obviously, the amplitudes Ψ^{sj}_{ml} are eigenfunctions of rotational Casimir invariants, i.e., essentially angular momentum and vorticity:

$$\|\mathbf{S}\|^2 \Psi^{sj}_{ml} = \|\hat{\mathbf{r}}\|^2 \Psi^{sj}_{ml} = \hbar^2 s(s+1) \Psi^{sj}_{ml},$$

$$\|\mathbf{V}\|^2 \Psi^{sj}_{ml} = \|\hat{\mathbf{t}}\|^2 \Psi^{sj}_{ml} = \hbar^2 j(j+1) \Psi^{sj}_{ml},$$

where, let us remind, in three dimensions we have the following expressions:

$$\|\mathbf{S}\|^2 = \mathbf{S}_1^2 + \mathbf{S}_2^2 + \mathbf{S}_3^2, \qquad \|\mathbf{V}\|^2 = \mathbf{V}_1^2 + \mathbf{V}_2^2 + \mathbf{V}_3^2,$$

and similarly for $\hat{\mathbf{r}}$, $\hat{\mathbf{t}}$. According to tradition, one uses such a basis that Ψ^{sj}_{ml} are also eigenfunctions of the third components of rotational generators,

$$\mathbf{S}_3\Psi^{sj}_{ml} = \hbar m\Psi^{sj}_{ml}, \qquad \mathbf{V}_3\Psi^{sj}_{ml} = \hbar l\Psi^{sj}_{ml}.$$

And, obviously, when the values n, k in the superposition (7.16) are kept fixed and we retain only the corresponding single term, for the resulting Ψ we have

$$\hat{\mathbf{r}}_3\Psi^{sj}_{\substack{ml\\nk}} = \hbar n\Psi^{sj}_{\substack{ml\\nk}}, \qquad \hat{\mathbf{t}}_3\Psi^{sj}_{\substack{ml\\nk}} = \hbar k\Psi^{sj}_{\substack{ml\\nk}}.$$

7.16 Representation matrices

In this way one is dealing with quantum states of well-definite values of magnitudes and third components of the angular momentum and vorticity. For the general n, the amplitudes $\Psi^{\alpha\beta}_{ml}$ have, of course, the well-definite values $(\hbar/i)^p C(\alpha, p)$, $(\hbar/i)^p C(\beta, p)$ of the Casimirs (7.13). And now it will be convenient to return for a while (at least in a formal way) to the general case of dimension n.

Let us again use the exponential formulas (7.9) for the elements of $W(\omega) \in \mathrm{SO}(V, g)$, $W(\omega) \in \mathrm{SO}(U, \eta)$, and just their simply numerical counterparts in $\mathrm{SO}(n, \mathbb{R})$,

$$W(\omega) = \exp\left(\frac{1}{2}\omega^a{}_b E^b{}_a\right),$$

where the basic matrices $E^b{}_a$ are simply given by

$$(E^b{}_a)^c{}_d = \delta^b{}_d\delta^c{}_a - \delta^{bc}\delta_{ad}$$

(just simply the numerical counterpart of (7.9) showing that one works just in $\mathbb{R}^n$ and $\mathrm{SO}(n, \mathbb{R})'$ not in V, U, $\mathrm{SO}(V, g)$, $\mathrm{SO}(U, \eta)$ basis-identified with the previous ones). And from now on let us again decide to work in purely analytical matrix form using orthonormal coordinates in V, U and identifying them with $\mathbb{R}^n$. Representation matrices $\mathcal{D}^\alpha$ are given by the following expresion:

$$\mathcal{D}^\alpha(\omega) = \exp\left(\frac{1}{2}\omega^a{}_b M^{\alpha b}{}_a\right),$$

where the $N(\alpha)\times N(\alpha)$ anti-hermitian matrices $M^{\alpha b}{}_{a}$ form irreducible representations of the Lie algebra $\mathrm{SO}(n,\mathbb{R})'$, thus, their commutation rules are identical with those for $E^{b}{}_{a}$.

Remark: For any $\alpha\in\Omega$ and for any pair of indices b, a, $M^{\alpha b}{}_{a}$ are just matrices not (b,a)-matrix elements of some M^{α}; let us notice in this connection that $a,b=\overline{1,n}$, whereas any $M^{\alpha b}{}_{a}$ is an $N(\alpha)\times N(\alpha)$-matrix. Obviously, when dealing with matrices $\mathcal{D}^{\alpha}(L)$, $\mathcal{D}^{\beta}(R)$, we must specialize the redundant "coordinates" $\omega^{a}{}_{b}$ to the ones parameterizing respectively the L- and R-terms of the two-polar splitting, writing, e.g.,

$$\mathcal{D}^{\alpha}\left(L(l)\right)=\exp\left(\frac{1}{2}l^{a}{}_{b}M^{\alpha b}{}_{a}\right),\qquad \mathcal{D}^{\beta}\left(R(r)\right)=\exp\left(\frac{1}{2}r^{a}{}_{b}M^{\beta b}{}_{a}\right).$$

For example, in three dimensions, where the pseudovector $\overline{k}$ may be used instead of the tensor $\omega^{b}{}_{a}$, i.e., $\mathcal{D}^{s}\left(W\left(\overline{k}\right)\right)=\exp\left(k^{a}M^{s}{}_{a}\right)$, we should write that

$$\mathcal{D}^{s}\left(L\left(\overline{l}\right)\right)=\exp\left(l^{a}M^{s}{}_{a}\right),\qquad \mathcal{D}^{j}\left(R\left(\overline{r}\right)\right)=\exp\left(r^{a}M^{j}{}_{a}\right),$$

where $M^{s}{}_{a}$ (s being non-negative integers and $a=1,2,3$) are basic $(2s+1)\times(2s+1)$, thus, odd-dimensional, anti-hermitian matrices representing in an irreducible way the Lie algebra $\mathrm{SO}(3,\mathbb{R})'$. Therefore,

$$[M^{s}{}_{a},M^{s}{}_{b}]=-\epsilon_{ab}{}^{c}M^{s}{}_{c},$$

and it is impossible to reduce simultaneously all $M^{s}{}_{a}$ to the block form. The apparently impossible even dimension $(2s+1)$ of $M^{s}{}_{a}$, thus, positive half-integer s will be an important point of our further analysis because $\mathrm{SO}(3,\mathbb{R})'$ (just as any $\mathrm{SO}(n,\mathbb{R})'$, $n\geq 3$) admits even-dimensional representations corresponding to the half-integer angular momentum, both for rigid and homogeneously deformable bodies.

7.17 Algebraic form of equations

Let us introduce Hermitian matrices

$$S^{\alpha a}{}_{b}=\frac{\hbar}{i}M^{\alpha a}{}_{b},$$

thus, for $n = 3$,

$$S^j{}_a = \frac{\hbar}{i} M^j{}_a,$$

and

$$\frac{1}{i\hbar}[S^j{}_a, S^j{}_b] = \epsilon_{ab}{}^c S^j{}_c.$$

These are standard well-known matrices, possible to be determined in purely algebraic terms, basing only on the commutation relations [37]. And it was just a surprise that there exist even-dimensional irreducible representations, experimentally compatible with the half-integer internal angular momentum spin. The $(2j+1)\times(2j+1)$ matrices S^j provide the quantum description of the angular momentum with the quantized magnitude $\hbar^2 j(j+1)$; j being a non-negative integer, or also a positive half-integer in the theory of fermionic objects.

The representation property of $\mathcal{D}^\alpha$, i.e.,

$$\mathcal{D}^\alpha(R_1 R_2) = \mathcal{D}^\alpha(R_1)\mathcal{D}^\alpha(R_2),$$

together with the definition of generators (7.8), (7.10) imply that certain obvious relationships which enable one to replace some differential operations and equations by algebraic ones. Namely, it is clear from the above formulas that

$$\begin{aligned}
\frac{\hbar}{i}\mathbf{\Lambda}^i{}_j(L)\mathcal{D}^\alpha(L) &= S^{\alpha i}{}_j\mathcal{D}^\alpha(L),\\
\frac{\hbar}{i}\mathbf{\Lambda}^A{}_B(R)\mathcal{D}^\beta(R) &= \mathcal{D}^\beta(R)S^{\beta A}{}_B,\\
\frac{\hbar}{i}\mathbf{\Upsilon}^a{}_b(L)\mathcal{D}^\alpha(L) &= \mathcal{D}^\alpha(L)S^{\alpha a}{}_b,\\
\frac{\hbar}{i}\mathbf{\Upsilon}^a{}_b(R)\mathcal{D}^\beta(R) &= S^{\beta a}{}_b\mathcal{D}^\beta(R);
\end{aligned}$$

expressions on the right-hand side meant, obviously, in the sense of the matrix multiplication.

In other words,

$$\mathbf{S}^i{}_j\Psi^{\alpha\beta} = S^{\alpha i}{}_j\Psi^{\alpha\beta}, \qquad \mathbf{V}^A{}_B\Psi^{\alpha\beta} = \Psi^{\alpha\beta}S^{\beta A}{}_B,$$

where $\Psi^{\alpha\beta}$ is an abbreviation for the $N(\alpha)\times N(\beta)$ matrices $\left[\Psi^{\alpha\beta}_{ml}\right]$ in (7.14) $\left(m = \overline{1, N(\alpha)},\ l = \overline{1, N(\beta)}\right)$. Obviously, everything is formally correct because $S^{\alpha i}{}_j$, $S^{\beta A}{}_B$ are, respectively, $N(\alpha)\times N(\alpha)$- and $N(\beta)\times N(\beta)$-matrices.

Let us stress once again that the indices (i,j), (A,B) label basic matrices within their sets; they do not refer to matrix elements.

From now on it will be convenient to write also (7.14), (7.16) in matrix terms,

$$\Psi^{\alpha\beta}(L,D,R) = \mathcal{D}^{\alpha}(L) f^{\alpha\beta}(D) \mathcal{D}^{\beta}(R^{-1});$$

obviously, the reduced amplitude $f^{\alpha\beta}(D)$ is an $N(\alpha) \times N(\beta)$-matrix depending only on deformation invariants $D_{aa} = Q^a = \exp(q^a)$.

Similarly, $\hat{\mathbf{r}}^a{}_b$ and $\hat{\mathbf{t}}^a{}_b$ act on $\Psi^{\alpha\beta}$ as follows:

$$\begin{aligned}
\hat{\mathbf{r}}^a{}_b \Psi^{\alpha\beta} &= \mathcal{D}^{\alpha}(L) S^{\alpha a}{}_b f^{\alpha\beta}(D) \mathcal{D}^{\beta}(R^{-1}), \\
\hat{\mathbf{t}}^a{}_b \Psi^{\alpha\beta} &= \mathcal{D}^{\alpha}(L) f^{\alpha\beta}(D) S^{\beta a}{}_b \mathcal{D}^{\beta}(R^{-1}).
\end{aligned}$$

Therefore, this action reduces simply to the action on the reduced amplitude $f^{\alpha\beta}$ only. It will be convenient to denote it as follows:

$$\overrightarrow{S^{\alpha a}}{}_b f^{\alpha\beta} := S^{\alpha a}{}_b f^{\alpha\beta}, \qquad \overleftarrow{S^{\beta a}}{}_b f^{\alpha\beta} := f^{\alpha\beta}(D) S^{\beta a}{}_b.$$

By assumption, the representations $\mathcal{D}^{\alpha}$ of $\mathrm{SO}(n,\mathbb{R})$ are irreducible, therefore, the matrices

$$C^{\alpha}(p) = \underbrace{S^{\alpha a}{}_b S^{\alpha b}{}_c \cdots S^{\alpha u}{}_w S^{\alpha w}{}_a}_{p \text{ factors}}$$

are proportional to the $N(\alpha) \times N(\alpha)$ identity matrices,

$$C^{\alpha}(p) = \left(\frac{\hbar}{i}\right)^p C(\alpha,p) I_{N(\alpha)}, \tag{7.17}$$

where the numbers $C(\alpha,p)$ are eigenvalues of the corresponding Casimir operators built of the generators of the left and right regular translations on $\mathrm{SO}(n,\mathbb{R})$, e.g.,

$$\mathbf{C}_{\mathrm{SO}(n,\mathbb{R})}(p) = \underbrace{\mathbf{\Lambda}^a{}_b \mathbf{\Lambda}^b{}_c \cdots \mathbf{\Lambda}^u{}_w \mathbf{\Lambda}^w{}_a}_{p \text{ factors}}.$$

So, finally, let us summarize the corresponding formulas for the physical case $n = 3$,

$$\|\mathbf{S}\|^2 \Psi^{sj} = \|\hat{\mathbf{r}}\|^2 \Psi^{sj} = \hbar^2 s(s+1) \Psi^{sj},$$

$$\|\mathbf{V}\|^2\Psi^{sj} = \|\hat{\mathbf{t}}\|^2\Psi^{sj} = \hbar^2 j(j+1)\Psi^{sj},$$
$$\mathbf{S}_a\Psi^{sj} = S^s{}_a\Psi^{sj}, \qquad \mathbf{V}_a\Psi^{sj} = \Psi^{sj}S^j{}_a,$$

in particular, in the standard representation,

$$\mathbf{S}_3\Psi^{sj}_{ml} = \hbar m\Psi^{sj}_{ml}, \qquad \mathbf{V}_3\Psi^{sj}_{ml} = \hbar l\Psi^{sj}_{ml}.$$

And just as for the general dimension value n, a little more complicated action of $\hat{\mathbf{r}}_a$, $\hat{\mathbf{t}}_a$ resulting in affecting the reduced $f(D)$-amplitudes,

$$\begin{aligned}\hat{\mathbf{r}}_a &: \quad f^{sj} \mapsto S^s{}_a f^{sj} = \overrightarrow{S^s}{}_a f^{sj},\\ \hat{\mathbf{t}}_a &: \quad f^{sj} \mapsto f^{sj}S^j{}_a = \overleftarrow{S^j}{}_a f^{sj}.\end{aligned}$$

In particular, again in the standard representation,

$$\begin{aligned}\hat{\mathbf{r}}_3 &: \quad \left[f^{sj}_{ml}\right] \mapsto \left[\hbar m f^{sj}_{ml}\right],\\ \hat{\mathbf{t}}_3 &: \quad \left[f^{sj}_{ml}\right] \mapsto \left[\hbar l f^{sj}_{ml}\right].\end{aligned}$$

7.18 Half-integer values of spin

And now we are ready to return to the problem of covering spaces and half-integer quantized angular momentum of rigid and deformable bodies. The problem of half-integer spin appeared in quantum mechanics due to experimental data concerning radiation spectra of atoms and molecules. Later on some theoretical work gave an evidence of the existence of even-dimensional representations of the Lie algebra $\mathrm{SO}(3,\mathbb{R})'$. Their exponentiation does not lead to representations of $\mathrm{SO}(3,\mathbb{R})$ but to representations of its universal covering group $\mathrm{SU}(2)$, roughly speaking to the double-valued representations of $\mathrm{SO}(3,\mathbb{R})$; in a sense to its projective representations.

As mentioned, there are some arguments that, in contrast to the current views, it need not be always the case that the wave amplitudes must be one-valued functions on the configuration space. In certain situations, when the homotopy group is finite, it seems to be sufficient that they are correctly defined on the universal covering manifold of the configuration space. A typical example is rigid body mechanics [3, 4, 5] and the mechanics of affinely-rigid bodies.

Let us begin with the general n-dimensional case, $n \geq 3$. The configuration space of the rigid body in n dimensions may be identified analytically with the special orthogonal group $\mathrm{SO}(n, \mathbb{R})$. For $n \geq 3$ the universal covering group $\mathrm{Spin}(n)$ is doubly-connected, and the corresponding canonical projection $\tau : \mathrm{Spin}(n) \to \mathrm{SO}(n, \mathbb{R})$ is $2 : 1$. The special case $n = 2$ is completely different, because then the homotopy group is $\mathbb{Z}$ just as in the covering of the circle $\mathrm{SO}(2, \mathbb{R}) \simeq \mathrm{U}(1)$ by $\mathbb{R}$. Therefore, in this case it does not seem possible to admit multi-valued wave functions, i.e., ones defined on $\mathbb{R}$. For $n = 3$ the covering group $\mathrm{Spin}(3)$ is isomorphic with the group of special (determinant-one) unitary matrices $\mathrm{SU}(2)$. For any $u \in \mathrm{SU}(2)$ the matrices $\pm u \in \mathrm{SU}(2)$ project under τ onto the same element of $\mathrm{SO}(3, \mathbb{R})$. Therefore, $\mathrm{Ker}\, \tau = \tau^{-1}(I_3) = \{I_2, -I_2\}$, i.e., the kernel consists of the unit 2×2 matrix I_2 and $-I_2$. Lie algebra $\mathrm{SU}(2)'$ consists of anti-hermitian traceless complex matrices, i.e., such ones that $\alpha^+ = \overline{\alpha}^T = -\alpha$, $\mathrm{Tr}\alpha = 0$. The most convenient choice of basis, commonly used in geometry and physics, is the following system:

$$e_a = \frac{1}{2i}\sigma_a, \qquad a = 1, 2, 3,$$

where σ_a are Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The basis $\{e_a\}$ exactly corresponds to the basis $\{E_a\}$ of $\mathrm{SO}(3, \mathbb{R})'$,

$$[e_a, e_b] = \epsilon_{ab}{}^c e_c.$$

Canonical coordinates of the first kind are given by components of the rotation vector $\overline{k} \in \mathbb{R}^3$,

$$u\left(\overline{k}\right) = \exp\left(k^a e_a\right) = \cos\frac{k}{2} I_2 - \frac{k^a}{k} \sin\frac{k}{2} i\sigma_a; \tag{7.18}$$

often one uses the relativistic convention σ_0 for I_2. This parameterization exactly corresponds to the usual rotation vector $\overline{k} \in \mathbb{R}^3$ in $\mathrm{SO}(3, \mathbb{R})'$, thus,

$$\tau\left(u\left(\overline{k}\right)\right) = R\left(\overline{k}\right) \quad \text{if} \quad k \leq \pi.$$

The main difference is that k, the magnitude of $\overline{k}$, runs over the doubled range $[0, 2\pi]$. Parameterization is singular in the sense that all points of the

limiting sphere $k = 2\pi$ in $\mathbb{R}^3$ represent the same point of SU(2), namely, $-I = u(2\pi \cdot \overline{n})$ for any versor $\overline{n} \in \mathbb{R}^3$, $(\overline{n} \cdot \overline{n}) = 1$. All points of $\mathbb{R}^3$ in the interior of the ball $k < 2\pi$ represent uniquely elements of SU(2); in particular, unlike the situation in $\mathrm{SO}(3, \mathbb{R})$ there is no antipodal identification for $k = \pi$, i.e., $u(\pi \cdot \overline{n}) \neq u(-\pi \cdot \overline{n})$. Obviously,

$$\tau^{-1}\left(R\left(\overline{k}\right)\right) = \left\{u\left(\overline{k}\right), -u\left(\overline{k}\right)\right\} = \left\{u\left(\overline{k}\right), u\left(\left(1 - \frac{2\pi}{k}\right)\overline{k}\right)\right\}.$$

The epimorphism $\tau : \mathrm{SU}(2) \to \mathrm{SO}(3, \mathbb{R})$ is given by the assignment: $\mathrm{SU}(2) \ni v \mapsto R \in \mathrm{SO}(3, \mathbb{R})$, where

$$vu\left(\overline{k}\right)v^{-1} = u\left(R\overline{k}\right).$$

For any non-negative integer or positive half-integer $s = 0, 1/2, 1, 3/2, \ldots$ $(s = j/2, j = 0, 1, 2, 3, \ldots)$ the Lie algebra $\mathrm{SU}(2)'$ does possess an irreducible representation of dimension $(2s + 1)$ (thus, all naturals admitted, not only the even ones) in terms of anti-hermitian matrices M^s, the basic ones ${M^s}_a$, $a = 1, 2, 3$, chosen so as to satisfy the following expression:

$$[{M^s}_a, {M^s}_b] = -{\epsilon_{ab}}^c {M^s}_c.$$

Obviously, the corresponding s-angular momentum matrices,

$${S^s}_a := \frac{\hbar}{i}{M^s}_a,$$

are Hermitian and do commute as follows:

$$\frac{1}{i\hbar}[{S^s}_a, {S^s}_b] = {\epsilon_{ab}}^c {S^s}_c.$$

Exponentiation

$$\mathcal{D}^s\left(u\left(\overline{k}\right)\right) = \mathcal{D}^s\left(\exp\left(k^a e_a\right)\right) := \exp\left(k^a {M^s}_a\right) = \exp\left(\frac{i}{\hbar}k^a {S^s}_a\right)$$

leads to unitary irreducible $(2s + 1)$-dimensional representations of SU(2). The matrices ${M^s}_a$, as mentioned, may be found on the purely algebraic basis of commutation relations [37], thus, for $n = 3$ they are explicitly known and standard. And so are $\mathcal{D}^s$, or more precisely their matrix elements as

the special functions on $\mathrm{SO}(3,\mathbb{R})$. This, by the way, is not the only possible method. Another one is solving of differential equations on the group manifold, or symmetrized Kronecker products of the basic representation of SU(2) by itself.

For non-negative integers s, i.e., for the odd values of $N(s) = (2s+1)$, $\mathcal{D}^s$ do not distinguish elements $\pm u \in \mathrm{SU}(2)$ τ-projecting onto the same elements $R \in \mathrm{SO}(3,\mathbb{R})$, so, as a matter of fact, they are representations of $\mathrm{SO}(3,\mathbb{R})$ (more precisely, they are τ-pull-backs of $\mathrm{SO}(3,\mathbb{R})$-representations to SU(2)), just the previously discussed $\mathcal{D}^s$,

$$\mathcal{D}^s(u) = \mathcal{D}^s(-u).$$

For the positive half-integers s, i.e., for the even values of $N(s) = (2s+1)$, $\mathcal{D}^s$ differ in sign at $u, -u \in \mathrm{SU}(2)$,

$$\mathcal{D}^s(u) = -\mathcal{D}^s(-u),$$

thus, they are non-projectable to $\mathrm{SO}(3,\mathbb{R})$. But for any fixed s, the squared moduli of the matrix elements or those of their linear combinations are pull-backs from $\mathrm{SO}(3,\mathbb{R})$, so the probabilistic interpretation of $\overline{\Psi}\Psi$ is not violated. The same holds when we superpose matrix elements of various $\mathcal{D}^s$, $\mathcal{D}^j$ but with the same parity of $(2s+1)$, $(2j+1)$, i.e., with the same "half-nesses" of s, j. But, in general, the probabilistic interpretation of $\overline{\Psi}\Psi$ is violated when different "half-nesses" of s, j are superposed. This is a toy model of the superselection between "fermionic" and "bosonic" situations. As we shall see, in a much more drastic form the problem appears in quantum mechanics of affinely-rigid bodies.

Having in view physical applications we do not consider the general case with $n > 3$, thus, our $\mathrm{Spin}(n)$ will be $\mathrm{Spin}(3) \simeq \mathrm{SU}(2)$. The planar problems $n = 2$ are of some physical relevance and will be briefly reviewed. However, the possibility of the half-integer spin does not appear then; at the same time, some other problems difficult for $n = 3$ become drastically simplified, just trivialized, for $n = 2$.

7.19 Affine spinors and polar decomposition

The configuration spaces of affinely-rigid body, i.e., roughly speaking (if translational motion is neglected) $\mathrm{GL}^+(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{R})$, are also doubly-

connected, and the problem of physically admissible two-valued wave functions also appears here. There is, however, some difficulty, namely, the intriguing and interesting fact that the universal covering groups $\overline{\mathrm{GL}^+(n,\mathbb{R})}$, $\overline{\mathrm{SL}(n,\mathbb{R})}$ are nonlinear, i.e., they do not possess faithful realizations in terms of finite matrices. This, by the way, was a reason for plenty of misunderstandings and vast time in field theory and quantum mechanics [30, 31]. The fact was known long ago to mathematicians, like, e.g., E. Cartan, but was forgotten and exotic for physicists. The nonlinearity of the mentioned coverings implies, in particular, that affine spinors (half-objects) must be either infinite-dimensional or ruled by nonlinear realizations of $\overline{\mathrm{GL}^+(n,\mathbb{R})}$, $\overline{\mathrm{SL}(n,\mathbb{R})}$ as abstract groups constructed with the help of loops in $\mathrm{GL}^+(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{R})$.

However, in quantum mechanics of affinely-rigid bodies the construction of multi-valued wave functions may be analytically overcome with the use of polar and two-polar splittings. Let us begin from the first one,

$$\varphi = UA = BU = \left(UAU^{-1}\right)U,$$

where $U \in \mathrm{SO}(n,\mathbb{R})$, and A, B are symmetric and positively definite (and in the case of $\mathrm{SL}(n,\mathbb{R})$ their determinants equal one). The splitting is unique and, because of this, $\mathrm{GL}^+(n,\mathbb{R})$ as a manifold (but not as a group) may be identified with the Cartesian products $\mathrm{SO}(n,\mathbb{R})\times\ \mathrm{Sym}^+(n,\mathbb{R})$ or $\mathrm{Sym}^+(n,\mathbb{R})\times\ \mathrm{SO}(n,\mathbb{R})$. The manifold $\mathrm{Sym}^+(n,\mathbb{R})$ is diffeomorphic with $\mathbb{R}^{n(n+1)/2}$ ($\mathbb{R}^6$ if $n=3$), therefore, the covering manifold may be identified with $\mathrm{Spin}(n)\times\ \mathrm{Sym}^+(n,\mathbb{R})$ or $\mathrm{Sym}^+(n,\mathbb{R})\times\ \mathrm{Spin}(n)$. In the physical case $n=3$, these splittings become

$$\mathrm{SU}(2)\times\mathrm{Sym}^+(3,\mathbb{R}) \simeq \mathrm{SU}(2)\times\mathbb{R}^6$$

or, alternatively,

$$\mathrm{Sym}^+(3,\mathbb{R})\times\mathrm{SU}(2) \simeq \mathbb{R}^6\times\mathrm{SU}(2).$$

Topological non-triviality is absorbed here by the factor $\mathrm{SO}(3,\mathbb{R})$ (in general by $\mathrm{SO}(n,\mathbb{R})$) and covered by $\mathrm{SU}(2)$ (in general by $\mathrm{Spin}(n)$). Therefore, the admissible multi-valued wave functions may be expanded as follows:

$$\Psi(u,A) = \sum_s \sum_{m=-s}^{s} \sum_{k=-s}^{s} C^s{}_{mk}(A)\mathcal{D}^s{}_{mk}(u),$$

where s are non-negative integers or positive half-integers, and the summation over m, k is performed in steps by one, $\mathcal{D}^s$ are matrices of irreducible unitary representations of SU(2), and (very important!) only half-integer or integer values of s may appear in a given expansion if $\overline{\Psi}\Psi$ is to be one-valued on $\mathrm{GL}^+(3,\mathbb{R})$, or, more precisely, if it is to be a pull-back from $\mathrm{GL}^+(3,\mathbb{R})$ to $\overline{\mathrm{GL}^+(3,\mathbb{R})}$. Therefore, in any admissible Ψ, ${C^s}_{mk} = 0$ either for all non-negative integer or for all positive half-integer s. To be completely rigorous, we would have to write

$$\Psi(u,A) = \sum_{\sigma=1}^{\infty}\sum_{\mu=0}^{\sigma}\sum_{\kappa=0}^{\sigma} C^{\frac{\sigma}{2}}{}_{(-\frac{\sigma}{2}+\mu),(-\frac{\sigma}{2}+\kappa)}(A)\mathcal{D}^{\frac{\sigma}{2}}{}_{(-\frac{\sigma}{2}+\mu),(-\frac{\sigma}{2}+\kappa)}(u)$$

for half-integer spin ("fermionic") situations or, respectively,

$$\Psi(u,A) = \sum_{s=0}^{\infty}\sum_{\mu=0}^{2s}\sum_{\kappa=0}^{2s} C^{s}{}_{(-s+\mu),(-s+\kappa)}(A)\mathcal{D}^{s}{}_{(-s+\mu),(-s+\kappa)}(u)$$

for integer spin ("bosonic") situations. These formulas are valid without any provisos, with summation over all indices meant in steps by one. If $\overline{\Psi}\Psi$ is to be one-valued probability distribution, then the superposing between indicated subspaces of function series is forbidden (a kind of superselection rule), and the admissible Hamiltonians must exclude any transitions between them; otherwise they are not well-defined on $\mathrm{L}^2(\mathrm{GL}^+(n,\mathbb{R}))$.

7.20 Two-polar splitting

It was said that the two-polar decomposition is maximally effective in problems on which we concentrate. Let us now describe the covering manifold $\overline{\mathrm{GL}^+(n,\mathbb{R})}$ and the corresponding two-valued wave functions on $\mathrm{GL}^+(n,\mathbb{R})$ in terms of the two-polar splitting. The non-uniqueness of the two-polar splitting of $\mathrm{GL}^+(n,\mathbb{R})$ was described briefly at the beginning of the chapter 6. Certain modifications are necessary when using this splitting for describing the covering $\overline{\mathrm{GL}^+(n,\mathbb{R})}$.

The elements of $\mathrm{GL}^+(n,\mathbb{R})$ were represented by the triplets $(L,D,R) \in \mathrm{SO}(n,\mathbb{R}) \times \mathbb{R}^n \times \mathrm{SO}(n,\mathbb{R})$ taken modulo certain identifications resulting from the fact that it was just the product $\varphi = LDR^{-1}$ not (L,D,R) itself that was a true configuration. Now, when describing $\overline{\mathrm{GL}^+(n,\mathbb{R})}$, we

must start from the triplets $(l, D, r) \in \mathrm{Spin}(n) \times \mathbb{R}^n \times \mathrm{Spin}(n)$, i.e., in the physical three-dimensional case $(l, D, r) \in \mathrm{SU}(2) \times \mathbb{R}^3 \times \mathrm{SU}(2)$. In this last case, l and r will be analytically described by the extended rotation vectors $\bar{l}, \bar{r} \in \mathbb{R}^3$ in the sense of (7.18) with $\bar{k}$ replaced respectively by $\bar{l}, \bar{r}$. Similarly, D is analytically represented by the variables $q^a = \ln D_{aa}$, and the dilatational degree of freedom by the centre $q = \left(q^1 + q^2 + q^3\right)/3$. As above, $\tau : \mathrm{Spin}(n) \to \mathrm{SO}(n, \mathbb{R})$ denotes the canonical projection ($2:1$ epimorphism). $\overline{K^+} := \tau^{-1}(K^+)$ is a $2n \cdot n!$-element subgroup of $\mathrm{Spin}(n)$; the group $K^+ \subset \mathrm{SO}(n, \mathbb{R})$ itself was defined in chapter 6. The manifold $M^{(n)}$ introduced also there is covered by $\overline{M^{(n)}}$, i.e., the subset of such triplets $\left(l; q^1, \ldots, q^n; r\right) \in \mathrm{Spin}(n) \times \mathbb{R}^n \times \mathrm{Spin}(n)$ that all q^i's are pairwise distinct. The subgroup $\overline{K^+}$ induces on $\overline{M^{(n)}}$ the transformation group $\overline{H^{(n)}}$ action of which is given by the following rule:

$$\left(l; q^1, \ldots, q^n; r\right) \mapsto \left(lu; q^{\pi_{\tau(u)}(1}, \ldots, q^{n)}; ru\right),$$

where $u \in \overline{K^+} \subset \mathrm{Spin}(n)$ (SU(2) if $n = 3$).

The corresponding generic part of $\overline{\mathrm{GL}^+(n, \mathbb{R})}$ (non-degenerate deformation tensors) is obtained as a quotient subset under the $\overline{H^{(n)}}$-action, i.e.,

$$Q^{(n)} \simeq \overline{M^{(n)}}/\overline{H^{(n)}}.$$

The situation becomes more complicated when some q^a's coincide, i.e., when the spectra of deformation tensors are degenerate. Let the symbols $Q^{(k;p_1,\ldots,p_k)} = \mathrm{GL}^{+(k;p_1,\ldots,p_k)}$, p_σ, $M^{(k;p_1,\ldots,p_k)}$, k, and $H^{(k;p_1,\ldots,p_k)}$ have the same meaning as in the beginning of chapter 6 where the two-polar splitting non-uniqueness was described. To describe the half-integer angular momentum, we must take the manifold $\overline{M^{(k;p_1,\ldots,p_k)}}$ consisting of triplets $\left(l; q^1, \ldots, q^n; r\right)$, where $l, r \in \mathrm{Spin}(n)$ and the system $\left(q^1, \ldots, q^n\right)$ is degenerate as above. Let $\overline{H^{(k;p_1,\ldots,p_k)}} \subset \mathrm{Spin}(n)$ denote the subgroup $\tau^{-1}\left(H^{(k;p_1,\ldots,p_k)}\right)$. The corresponding manifolds of degenerate configurations are given by the quotient subsets

$$\overline{M^{(k;p_1,\ldots,p_k)}}/\overline{H^{(k;p_1,\ldots,p_k)}} \tag{7.19}$$

in the sense of the action

$$\left(l; q^1, \ldots, q^n; r\right) \mapsto \left(lu; q^{\pi_{\tau(u)}(1}, \ldots, q^{n)}; ru\right); \tag{7.20}$$

obviously, u runs over $\overline{H^{(k;p_1,\ldots,p_k)}}$.

The admissibly multi-valued wave functions on $\mathrm{GL}^+(n,\mathbb{R})$, i.e., the ones that are one-valued on $\overline{\mathrm{GL}^+(n,\mathbb{R})}$, are represented by complex amplitudes on $\mathrm{Spin}(n)\times\mathbb{R}^n\times\mathrm{Spin}(n)$ which are invariant under the above actions of $\overline{H^{(k;p_1,\ldots,p_k)}}$ (7.20), i.e., are projectable onto the resulting quotients (7.19).

7.21 Three-dimensional physical case

Let us now concentrate on the special case $n = 3$, both the practically important one and at the same time reducible in a sense to the classical Wigner results [55, 86, 87].

All the former expressions concerning function series, eigenequations, etc. remain generally true with the following changes: half-integer quantum numbers s, j, m, k, l, n, etc. are admissible, and certain new complications appear concerning the non-distinguishability of triplets $\left(l;q^1,q^2,q^3;r\right)$ by wave functions representants. In particular, some correlation appears between "half-nesses" of the quantum numbers s, j (spin and vorticity) in physically acceptable function series. Obviously, this is based on the assumption (true or not?) that the wave functions Ψ may be multi-valued, but their moduli $|\Psi|$ must be one-valued in accordance with the statistical interpretation of $\overline{\Psi}\Psi$.

It is known that $\mathcal{D}^j(u) = \pm\mathcal{D}^j(-u)$, $u\in\mathrm{SU}(2)$, depending, respectively, on whether j is integer or half-integer. Therefore, the expansions (7.15), (7.16) remain valid for half-integer spin and vorticity, thus, within the framework $\mathrm{SU}(2)\times\mathbb{R}^3\times\mathrm{SU}(2)$ provided that some care is taken what concerns the superposition structure, more precisely, the (s,j) correlation. So, formally, we can rewrite (7.15), (7.16) as follows:

$$\Psi(u,D,v) = \sum_{s,j}\sum_{m,n=-s}^{s}\sum_{k,l=-j}^{j}\mathcal{D}^s_{mn}(u)f^{sj}_{\substack{nk\\ml}}(D)\mathcal{D}^j_{kl}(v^{-1}), \quad (7.21)$$

$$\Psi^{sj}_{ml}(u,D,v) = \sum_{n=-s}^{s}\sum_{k=-j}^{j}\mathcal{D}^s_{mn}(u)f^{sj}_{nk}(D)\mathcal{D}^j_{kl}(v^{-1}), \quad (7.22)$$

with the following descriptive comments. Summation over (s,j) in (7.21) or the choice of particular (s,j) in (7.22) is extended over non-negative integers or positive half-integers, but in such a way that either (s,j) are

simultaneously integers or simultaneously half-integers. In other words, $f^{sj}(q^1,q^2,q^3)\equiv 0$ if the number $(j-s)$ is half-integer, i.e., always the summation will be extended over such pairs (s,j) in (7.21) or the values of (s,j) will be chosen in (7.22)in such a way that $(j-s)$ will be an integer number. The quantum numbers (m,n), (k,l) in (7.21) run over the ranges from $-s$ to s and from $-j$ to j in integer jumps.

Just as for integer pairs (s,j), we will use $(2s+1)\times(2j+1)$ rectangular matrices $\Psi^{sj}=\left[\Psi^{sj}_{ml}\right]$, $f^{sj}=\left[f^{sj}_{nk}\right]$, where

$$\Psi^{sj}\left(u;q^1,q^2,q^3;v\right)=\mathcal{D}^s(u)f^{sj}\left(q^1,q^2,q^3\right)\mathcal{D}^j(v^{-1}).$$

As mentioned, f^{sj} vanishes identically as a function of q^a's when $(j-s)$ is half-integer. The matrix elements of Ψ^{sj} with integer values of $(j-s)$ may be arbitrarily superposed, and this correlation is a necessary condition if Ψ, Ψ^{sj} are to be well-defined on $\overline{\mathrm{GL}(3,\mathbb{R})}$ not only on the auxiliary manifold $\mathrm{SU}(2)\times\mathbb{R}^3\times\mathrm{SU}(2)$. If $\overline{\Psi}\Psi$ is to be projectable onto $\mathrm{GL}(3,\mathbb{R})$ (statistical interpretation), then we may superpose only terms with half-integer (s,j) or integer (s,j) separately.

One is used to avoid in mathematical texts the descriptive literature-like comments as above, however, sometimes the purely formula-based presentation becomes more obscure. It is just the case here, especially when we wish to retain the traditional notation used in the theory of angular momentum. So, for example, avoiding words following the formulas (7.21), (7.22) would be panished by the following rather obscure expressions:

$$\Psi(u,D,v)=\Psi_1(u,D,v)+\Psi_2(u,D,v)=\sum_{\sigma,\iota=1}^{\infty}$$

$$\sum_{\mu,\nu=0}^{\sigma}\sum_{\kappa,\lambda=0}^{\iota}\mathcal{D}^{\frac{\sigma}{2}}_{\left(-\frac{\sigma}{2}+\mu\right),\left(-\frac{\sigma}{2}+\nu\right)}(u)f^{\frac{\sigma}{2},\frac{\iota}{2}}_{\substack{(-\frac{\sigma}{2}+\nu),(-\frac{\iota}{2}+\kappa)\\(-\frac{\sigma}{2}+\mu),(-\frac{\iota}{2}+\lambda)}}(D)\mathcal{D}^{\frac{\iota}{2}}_{\left(-\frac{\iota}{2}+\kappa\right),\left(-\frac{\iota}{2}+\lambda\right)}(v^{-1})$$

$$+\sum_{s,j=0}^{\infty}\sum_{\mu,\nu=0}^{2s}\sum_{\kappa,\lambda=0}^{2j}\mathcal{D}^{s}_{(-s+\mu),(-s+\nu)}(u)f^{sj}_{\substack{(-s+\nu),(-j+\kappa)\\(-s+\mu),(-j+\lambda)}}(D)\mathcal{D}^{j}_{(-j+\kappa),(-j+\lambda)}(v^{-1}).$$

The first term Ψ_1 contains contributions with half-integer spin and vorticity (simultaneously), the second one Ψ_2 involves only integer quantized values of both. And this fact again means that Ψ is well-defined on $\overline{\mathrm{GL}(3,\mathbb{R})}$ not only on $\mathrm{SU}(2)\times\mathbb{R}^3\times\mathrm{SU}(2)$. But if $\overline{\Psi}\Psi$ is to be well-defined on $\mathrm{GL}(3,\mathbb{R})$ itself, then only Ψ_1,Ψ_2-terms are separately admissible without being superposed.

7.22 Reduction to Cartan subgroup

Matrix elements of irreducible representations have important well-investigated properties which enable one to algebraize a good deal of differential equations problems and to perform an effective reduction of the quantum dynamics. Roughly speaking, this is reduction to the Cartan subgroup of $\mathrm{GL}(n,\mathbb{R})$, i.e., to its maximal Abelian subgroup. This is just the group of diagonal matrices, i.e., degrees of freedom parameterized by deformation invariants $q^1,\dots,q^n$. This reduction from n^2 to n degrees of freedom is possible for geodetic problems, for dilatationally-stabilized problems (i.e., essentially for geodetic problems on $\mathrm{SL}(n,\mathbb{R})$) and, more generally, for doubly isotropic models when the potential energy is non-trivial but depends only on the deformation invariants, i.e., it has the form $V\left(q^1,\dots,q^n\right)$. Let us remind that in this sense quantum mechanics of affine bodies is "simpler" than the classical one where for $n>2$ there is no simple way of reducing equations of motion to the Cartan subgroup.

It is convenient to start again with the general n, and later on to restrict ourselves to the special cases $n=2,3$. Due to the standard orthogonality properties of ${\mathcal{D}^{\alpha}}_{mn}$, the scalar product of wave functions Ψ may be reduced to one for the amplitudes $f^{\alpha\beta}$ depending only on deformation invariants, i.e.,

$$<\Psi_1|\Psi_2>=\sum_{\alpha,\beta\in\Omega}\frac{1}{N(\alpha)N(\beta)}\int\sum_{n,m=1}^{N(\alpha)}\sum_{k,l=1}^{N(\beta)}\overline{f_1}^{\alpha\beta}_{\substack{nk\\ml}}f_2{}^{\alpha\beta}_{\substack{nk\\ml}}Pdq^1\cdots dq^n,$$

where, let us remind, the weight P is given by the following expression:

$$P\left(q^1,\dots,q^n\right)=\prod_{i\neq j}\left|\mathrm{sh}(q^i-q^j)\right|.$$

If we fix the labels α, β, m, l ("good" quantum numbers for doubly-isotropic problems) and consider the simplified $N(\alpha)\times N(\beta)$-matrix amplitudes,

$$\Psi^{\alpha\beta}\left(L;q^1,\dots,q^n;R\right)=\mathcal{D}^{\alpha}(L)f^{\alpha\beta}\left(q^1,\dots,q^n\right)\mathcal{D}^{\beta}\left(R^{-1}\right),$$

then the scalar product reduces to

$$<\Psi_1^{\alpha\beta}|\Psi_2^{\alpha\beta}>=\frac{1}{N(\alpha)N(\beta)}$$

$$\int \mathrm{Tr}\left(f_1^{\alpha\beta+}\left(q^1,\ldots,q^n\right) f_2^{\alpha\beta}\left(q^1,\ldots,q^n\right)\right) P\left(q^1,\ldots,q^n\right) dq^1\cdots dq^n,$$

where, obviously, $f_1^{\alpha\beta+}$ denotes the Hermitian conjugate of the matrix $f_1^{\alpha\beta}$.

Obviously, for the general expansion (7.12) the corresponding formula involves the summation over α, β, and the multiplication of reduced amplitudes and trace operation meant in the sense of two-matrices with the entries labelled by two-indices $f^{\alpha\beta}_{\substack{nk\\ml}}$, i.e.,

$$< \Psi_1|\Psi_2 >= \sum_{\alpha,\beta\in\Omega} \frac{1}{N(\alpha)N(\beta)}$$

$$\int \mathrm{Tr}\left(f_1^{\alpha\beta+}\left(q^1,\ldots,q^n\right) f_2^{\alpha\beta}\left(q^1,\ldots,q^n\right)\right) P\left(q^1,\ldots,q^n\right) dq^1\cdots dq^n,$$

For the sake of completeness, let us write explicitly

$$\mathrm{Tr}\left(f_1^{\alpha\beta+} f_2^{\alpha\beta}\right) = \sum_{n,m=1}^{N(\alpha)} \sum_{k,l=1}^{N(\beta)} \overline{f_1}^{\alpha\beta}_{\substack{nk\\ml}} f_2{}^{\alpha\beta}_{\substack{nk\\ml}}.$$

When we consider the class of problems with α, β, m, l fixed once for all, then one can avoid the divisor $N(\alpha)N(\beta)$, with the proviso of being careful with the normalization of amplitudes so as not to violate the statistical interpretation.

In certain problems it may be convenient to avoid the phase factor P in the above expressions for the scalar product. To achieve this one should introduced rescaled amplitudes given by the following matrices:

$$g^{\alpha\beta} := \sqrt{P} f^{\alpha\beta}.$$

Then the factor P disappears from the above formulas, $f^{\alpha\beta}$ becomes replaced by $g^{\alpha\beta}$, and everything else remains as previously.

7.23 Metric tensors and arc elements

Essentially everything said above remains valid when discussing the half-integer angular momentum. Orthogonal groups $\mathrm{SO}(n,\mathbb{R})$ in the two-polar decomposition are then replaced by their coverings $\mathrm{Spin}(n)$, but it does

not change anything in local analytical expressions. Technically, the only change is that the range of group parameters changes. And where for different parameter values the corresponding elements of $\mathrm{SO}(n,\mathbb{R})$ were identical, in $\mathrm{Spin}(n)$ they are different. It was described above in some details for $\mathrm{SO}(3,\mathbb{R})$ and its covering $\mathrm{Spin}(n)=\mathrm{SU}(2)$, where the main analytical novelty was replacing the range $[0,\pi]$ for the rotation vector magnitude k with $[0,2\pi]$. All analytical formulas remain formally the same, e.g., those for the generators of left and right regular translations $\mathbf{\Lambda}_a$, $\mathbf{\Upsilon}_a$. The metric Killing tensors on $\mathrm{SO}(3,\mathbb{R})$ and $\mathrm{SU}(2)$ normalized to be δ_{ij} in $\overline{k}$-coordinates at the group identity (thus, differing by the minus one-half factor in comparison with the general Lie-algebraic definition), i.e.,

$$\Gamma(a,b)=-\frac{1}{2}\mathrm{Tr}(ab),\qquad \Gamma(a,b)=-2\mathrm{Tr}(ab)$$

respectively, on $\mathrm{SO}(3,\mathbb{R})$ and $\mathrm{SU}(2)$, in both cases they are analytically given by the same formula:

$$\Gamma_{ab}=\frac{4}{k^2}\sin^2\frac{k}{2}\delta_{ab}+\left(1-\frac{4}{k^2}\sin^2\frac{k}{2}\right)\frac{k_ak_b}{k^2}.$$

In other words, the corresponding arc element is as follows:

$$ds^2=\Gamma_{ab}dk^adk^b=dk^2+4\sin^2\frac{k}{2}\left(d\vartheta^2+\sin^2\vartheta d\varphi^2\right).$$

Obviously, this metric is conformally flat, e.g., defining new coordinates

$$\overline{r}=(a/k)\mathrm{tg}(k/4)\overline{k},\qquad a>0,$$

we obtain that

$$ds^2=\frac{16a^2}{a^2+r^2}\left(dr^2+r^2\left[d\vartheta^2+\sin^2\vartheta d\varphi^2\right]\right),$$

where the second factor is just the arc element in Euclidean $\mathbb{R}^3$ expressed in terms of spherical coordinates. This is the conformal mapping of $\mathrm{SU}(2)$ onto $\mathbb{R}^3$ if we consider the total range $r\in[0,\infty]$. It is interesting that $r\in[0,a]$ on $\mathrm{SO}(3,\mathbb{R})$. This is also some kind of arguments that $\mathrm{SO}(3,\mathbb{R})$ is somehow "imperfect" in comparison with its universal covering $\mathrm{SU}(2)$.

The Haar measure μ in both cases is given by

$$d\mu\left(\overline{k}\right) = \frac{4}{k^2}\sin^2\frac{k}{2}d_3\overline{k} = 4\sin^2\frac{k}{2}\sin\vartheta dk d\vartheta d\varphi$$

if we wish its weight function to be equal one in $\overline{k}$-coordinates at the unit element $\left(\overline{k}=0\right)$. But if we wish, as we often do, to normalize the total measure of the compact group to unity, then both cases will differ by a constant factor.

7.24 Quantizing affine-affine kinetic energies

One can show after some calculations that the operator $\mathbf{T}_{\rm int}^{\rm aff-aff}$ of kinetic energy invariant under both spatial and material affine transformation is as follows:

$$\begin{aligned}\mathbf{T}_{\rm int}^{\rm aff-aff} &= -\frac{\hbar^2}{2A}\mathbf{D} + \frac{\hbar^2 B}{2A(A+nB)}\frac{\partial^2}{\partial q^2} \\ &+ \frac{1}{32A}\sum_{a,b}\frac{(\mathbf{M}^a{}_b)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}} - \frac{1}{32A}\sum_{a,b}\frac{(\mathbf{N}^a{}_b)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}},\end{aligned} \tag{7.23}$$

where A, B are constants as previously in classical formulas,

$$\mathbf{M}^a{}_b = -\hat{\mathbf{r}}^a{}_b - \hat{\mathbf{t}}^a{}_b, \qquad \mathbf{N}^a{}_b = \hat{\mathbf{r}}^a{}_b - \hat{\mathbf{t}}^a{}_b,$$

(cf. (7.5), (7.6)), and

$$\mathbf{D} = \frac{1}{P}\sum_a \frac{\partial}{\partial q^a}P\frac{\partial}{\partial q^a} = \sum_a \frac{\partial^2}{\partial (q^a)^2} + \sum_a \frac{\partial \ln P}{\partial q^a}\frac{\partial}{\partial q^a}$$

(every differentiation operator acts on everything on the right of it), P is the previously introduced weight factor.

It is seen that this is almost the previously used classical formula with classical canonical quantities, e.g., $\hat{\rho}^a{}_b$, $\hat{\tau}^a{}_b$ replaced by the corresponding operators $\hat{\mathbf{r}}^a{}_b$, $\hat{\mathbf{t}}^a{}_b$. There is, however, some difference and possibility of an easy mistake in the sector of (q^a, p_a)-variables. Namely, the term involving differentiation with respect to q^a is not, as it might be expected, the usual $\mathbb{R}^n$-Laplace operator in q^a variables, although it contains such a term. Let

us observe that in the $\varphi = LDR^{-1}$-representation the $\partial/\partial q^a$ operators act only on the $f^{\alpha\beta}$ amplitude, whereas $\hat{\mathbf{r}}^a{}_b$, $\hat{\mathbf{t}}^a{}_b$ act only, respectively, on the L- and R-variables. Therefore, there is no problem of ordering of operators in $\mathbf{T}_{\text{int}}^{\text{aff}-\text{aff}}$. One could get rid off the first derivatives of Ψ with respect to q^a by the substitution which was already used within a slightly different context, namely,

$$\varphi = \sqrt{P}\Psi.$$

The action of the last three terms in (7.23) on φ is exactly as that on Ψ because $\partial/\partial q^a$, $\mathbf{M}^a{}_b$, $\mathbf{N}^a{}_b$ do not act on $(q^a - q^b)$-quantities of which P is built; roughly speaking, the $\sqrt{P}$ is "transparent" for these operators. It is no longer the case with the $\mathbf{D}$-term, both in the good and in the bad senses. Namely, the action of $-(\hbar^2/2A)\mathbf{D}$ on Ψ is represented by the action of the following operator $-(\hbar^2/2A)\widetilde{\mathbf{D}}$ on φ:

$$-\frac{\hbar^2}{2A}\widetilde{\mathbf{D}} = -\frac{\hbar^2}{2A}\sum_a \frac{\partial^2}{\partial(q^a)^2} + \widetilde{\mathbf{V}},$$

where $\widetilde{\mathbf{V}}$ is the following artificial potential term:

$$\widetilde{\mathbf{V}} = -\frac{\hbar}{2A}\frac{1}{P^2} + \frac{\hbar^2}{4A}\frac{1}{P}\sum_a \left(\frac{\partial P}{\partial q^a}\right)^2.$$

In other words,

$$\widetilde{\mathbf{D}}\varphi = \sqrt{P}\mathbf{D}\Psi.$$

There are no first derivatives of φ with respect to q^a, and the differential action is given by the usual $\mathbb{R}^n$-Laplace operator, just as in mechanics of n q^a-particles on $\mathbb{R}$. But this simplification is only seeming one because, if $n > 2$, it is completely destroyed by the "potential" $\widetilde{\mathbf{V}}$. Obviously, in realistic problems concerning deformable objects Hamiltonian should also contain dilatation-stabilizing potential, i.e.,

$$\mathbf{H} = \mathbf{T}_{\text{int}}^{\text{aff}-\text{aff}} + \mathbf{V}(q).$$

Although such simple $\mathrm{SL}(n,\mathbb{R})$-geodetic models may successfully describe elastic vibrations, some more general isotropic potentials $V\left(q^1,\ldots,q^n\right)$ are also acceptable and compatible with the above description.

7.25 Metric-affine and affine-metric models

Quantizing metric-affine and affine-metric kinetic energies we obtain, respectively, the following operators:

$$\begin{aligned}
\mathbf{T}^{\mathrm{met-aff}}_{\mathrm{int}} &= -\frac{\hbar^2}{2\alpha}\mathbf{D} - \frac{\hbar^2}{2\beta}\frac{\partial^2}{\partial q^2} \\
&+ \frac{1}{32\alpha}\sum_{a,b}\frac{(\mathbf{M}^a{}_b)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}} - \frac{1}{32\alpha}\sum_{a,b}\frac{(\mathbf{N}^a{}_b)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}} + \frac{1}{2\mu}\|\mathbf{S}\|^2, \\
\mathbf{T}^{\mathrm{aff-met}}_{\mathrm{int}} &= -\frac{\hbar^2}{2\alpha}\mathbf{D} - \frac{\hbar^2}{2\beta}\frac{\partial^2}{\partial q^2} \\
&+ \frac{1}{32\alpha}\sum_{a,b}\frac{(\mathbf{M}^a{}_b)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}} - \frac{1}{32\alpha}\sum_{a,b}\frac{(\mathbf{N}^a{}_b)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}} + \frac{1}{2\mu}\|\mathbf{V}\|^2,
\end{aligned}$$

with the same meaning of operator symbols as above and the same relationship between inertial constants (α, β, μ) and the primary ones (I, A, B) as previously in (4.26), (4.27).

7.26 Potential case

As mentioned above, for Hamiltonians $\mathbf{H} = \mathbf{T} + \mathbf{V}$ with some dilatation-stabilizing potentials $V(q)$, or more generally, with some doubly-isotropic potentials $V(q^1, \ldots, q^n)$, the action of operators $\mathbf{M}^a{}_b$ and $\mathbf{N}^a{}_b$ become algebraic and standard, and the stationary Schrödinger equation, i.e., energy eigenproblem

$$\mathbf{H}\Psi = E\Psi,$$

splits into family of eigenproblems for the amplitudes $f^{\alpha\beta}$; they are partial differential equations involving q^a-variables only:

$$\mathbf{H}^{\alpha\beta}f^{\alpha\beta} = E^{\alpha\beta}f^{\alpha\beta},$$

where $f^{\alpha\beta}$ for any $\alpha, \beta \in \Omega$ is an $N(\alpha) \times N(\beta)$ matrix depending on $q^1, \ldots, q^n$. In a consequence of the double (spatial and material) isotropy, this problem is $N(\alpha) \times N(\beta)$-fold degenerate, i.e., for every component of $f^{\alpha\beta}$ there exists an $N(\alpha) \times N(\beta)$-dimensional subspace of solutions. Let us remind that in the primary symbols $f^{\alpha\beta}_{\substack{nk\\ml}}$ the indices m, l just label the

degeneracy of solutions for every $f^{\alpha\beta}_{nk}$. $\mathbf{H}^{\alpha\beta}$ is an $N(\alpha)\times N(\beta)$-matrix of second-order differential operators,

$$\mathbf{H}^{\alpha\beta}=\mathbf{T}^{\alpha\beta}+\mathbf{V},$$

where $\mathbf{V}$ denotes a dilatation-stabilizing or general doubly-isotropic potential, and $\mathbf{T}^{\alpha\beta}$ denotes the kinetic energy operator. It is one of the previous ones restricted to the corresponding (α,β)-subspace.

Therefore, for the affine-affine, metric-affine, and affine-metric models we have, respectively,

$$\begin{aligned}
\mathbf{T}^{\alpha\beta}f^{\alpha\beta} &= -\frac{\hbar^2}{2A}\mathbf{D}f^{\alpha\beta}+\frac{1}{32A}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}{}^{a}{}_{b}-\overrightarrow{S^{\alpha}}{}^{a}{}_{b}\right)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}}f^{\alpha\beta} \qquad (7.24)\\
&- \frac{1}{32A}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}{}^{a}{}_{b}+\overrightarrow{S^{\alpha}}{}^{a}{}_{b}\right)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}}f^{\alpha\beta}+\frac{\hbar^2 B}{2A(A+nB)}\frac{\partial^2}{\partial q^2}f^{\alpha\beta},\\
\mathbf{T}^{\alpha\beta}f^{\alpha\beta} &= -\frac{\hbar^2}{2\alpha}\mathbf{D}f^{\alpha\beta}-\frac{\hbar^2}{2\mu}C(\alpha,2)f^{\alpha\beta}+\frac{1}{32\alpha}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}{}^{a}{}_{b}-\overrightarrow{S^{\alpha}}{}^{a}{}_{b}\right)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}}f^{\alpha\beta}\\
&- \frac{1}{32\alpha}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}{}^{a}{}_{b}+\overrightarrow{S^{\alpha}}{}^{a}{}_{b}\right)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}}f^{\alpha\beta}-\frac{\hbar^2}{2\beta}\frac{\partial^2}{\partial q^2}f^{\alpha\beta}, \qquad (7.25)\\
\mathbf{T}^{\alpha\beta}f^{\alpha\beta} &= -\frac{\hbar^2}{2\alpha}\mathbf{D}f^{\alpha\beta}-\frac{\hbar^2}{2\mu}C(\beta,2)f^{\alpha\beta}+\frac{1}{32\alpha}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}{}^{a}{}_{b}-\overrightarrow{S^{\alpha}}{}^{a}{}_{b}\right)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}}f^{\alpha\beta}\\
&- \frac{1}{32\alpha}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}{}^{a}{}_{b}+\overrightarrow{S^{\alpha}}{}^{a}{}_{b}\right)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}}f^{\alpha\beta}-\frac{\hbar^2}{2\beta}\frac{\partial^2}{\partial q^2}f^{\alpha\beta}, \qquad (7.26)
\end{aligned}$$

where the meaning of Casimir eigenvalues $C(\alpha,2)$, $C(\beta,2)$ like in (7.17). The constants α, β, μ are exactly as previously; do not confuse them with labels α, β at $f^{\alpha\beta}$. In the physical case $n=3$, $\alpha=s=0,1/2,1,\ldots\in\mathbb{N}/2\cup\{0\}$ and similarly $\beta=j=0,1/2,1,\ldots\in\mathbb{N}/2\cup\{0\}$ assuming that the half-integer values of angular momentum and vorticity are admitted. Otherwise we would have $s,j\in\mathbb{N}\cup\{0\}$. Obviously, in this case $C(s,2)=-s(s+1)$,

$C(j,2) = -j(j+1)$, and the additional constants in the last two formulas are simply $(\hbar^2/2\mu)s(s+1)$, $(\hbar^2/2\mu)j(j+1)$, expressions close to the heart of any physicist. Let us stress that, even if half-integers are admitted, there is a restriction that $(j-s)$ must be integer, i.e., j and s have the same "halfness". In any case, it must be so if wave functions are to be well-defined on $\overline{\mathrm{GL}(3,\mathbb{R})}$ not only on the "artificial" configuration space $\mathrm{SU}(2)\times\mathbb{R}^3\times\mathrm{SU}(2)$. If they are to be statistically interpretable in $\mathrm{GL}(3,\mathbb{R})$ itself, then only the terms with half-integer (s,j) or integer (s,j) may be separately superposed, no mutual superposition admissible (although some blasphemic doubts may be raised against this superselection, i.e., against statistical interpretation in $\mathrm{GL}(3,\mathbb{R})$).

In three dimensions the above-mentioned additional terms $(\hbar^2/2\mu)s(s+1)$, $(\hbar^2/2\mu)j(j+1)$ seem to be physically interesting and, at least qualitatively, compatible with some experimental data. It is so as if the doubly affine background (affine invariance in space and in the body) was responsible for some fundamental part of the spectra, which later on, the more the μ is smaller, splits due to some internal rotations. The term $(\hbar^2/2\mu)s(s+1)$ is physically intuitive and classically corresponds to the situation when in the system some regime of rigid rotations was established after time of transition processes. But, perhaps, $(\hbar^2/2\mu)j(j+1)$ appearing in the affine-metrical model is even more interesting. Being a formal analogue of certain aspects of angular momentum, it is not angular momentum and may be perhaps semiclassically related to the isotopic spin or similar internal quantities ruled by SU(2) and appearing in nuclear and elementary particle physics.

Remark: just as previously, the terms with the first-order derivatives of $f^{\alpha\beta}$ with respect to q^a may be avoided by the substitution

$$g^{\alpha\beta} := \sqrt{P} f^{\alpha\beta},$$

which was also used for simplifying the scalar product. But then again the artificial potential $\mathbf{V}$ appears in all reduced Schrödinger equations.

7.27 Combine models and their possible use

By the way, one can have both things, i.e., the terms $(\hbar^2/2\mu)s(s+1)$ and $(\hbar^2/2\mu)j(j+1)$ terms. For this purpose we would have to use the kinetic

energy consisting of four terms:

$$T_{\rm int} = \frac{I_1}{2} g_{ik} g^{jl} \Omega^i{}_j \Omega^k{}_l + \frac{I_2}{2} \eta_{KL} \eta^{MN} \hat{\Omega}^K{}_M \hat{\Omega}^L{}_N + \frac{A}{2} \hat{\Omega}^K{}_L \hat{\Omega}^L{}_K + \frac{B}{2} \hat{\Omega}^K{}_K \hat{\Omega}^L{}_L,$$

where, obviously, the last two terms might be as well written as follows:

$$\frac{A}{2} \Omega^i{}_j \Omega^j{}_i + \frac{B}{2} \Omega^i{}_i \Omega^j{}_j.$$

In matrix language, using Cartesian coordinates $g_{ik} =_* \delta_{ik}$, $\eta_{AB} =_* \delta_{AB}$, we would simply write that

$$\begin{aligned} T_{\rm int} &= \frac{I_1}{2} {\rm Tr}(\Omega^T \Omega) + \frac{I_2}{2} {\rm Tr}(\hat{\Omega}^T \hat{\Omega}) + \frac{A}{2} {\rm Tr}(\hat{\Omega}^2) + \frac{B}{2} ({\rm Tr}\hat{\Omega})^2 \quad (7.27) \\ &= \frac{I_1}{2} {\rm Tr}(\Omega^T \Omega) + \frac{I_2}{2} {\rm Tr}(\hat{\Omega}^T \hat{\Omega}) + \frac{A}{2} {\rm Tr}(\Omega^2) + \frac{B}{2} ({\rm Tr}\Omega)^2. \end{aligned}$$

But now some reproach might be raised that, doing as above, we forget our primary motivation concerning the dynamical $\mathrm{GL}(n, \mathbb{R})$-invariance and return to models which are only orthogonally invariant (geometrically speaking, $\mathrm{O}(V, g)$- and $\mathrm{O}(U, \eta)$-invariant), and it is again only pure kinematics that is ruled by affine group. This would be true, and we indeed do not insist on the above model. Let us notice, however, that this model, having still high dynamical symmetry, may also work as a purely geodetic model encoding a kind of elastic bounded vibrations without any extra introduced potential. Moreover, due to the lack of dilatational invariance, it is not excluded (we are not yet sure; this is a conjecture) that even dilatation-stabilizing potentials would not be necessary.

7.28 Doubly isotropic d'Alembert models

The above remarks about models (7.27) again put our attention on the doubly isotropic "d'Alembert" models of classical kinetic energy (2.1), i.e., (4.4) and (4.5) with $J^{KL} = I\eta^{KL}$. The corresponding kinetic part of the classical kinetic Hamiltonian $\mathcal{T}^{\rm d.A}_{\rm int}$ was given by (6.10) with the same meaning of $M^a{}_b$, $N^a{}_b$ as above, $Q^a = D_{aa}$ are diagonal elements of D, and P_a are canonical momenta conjugate to Q^a. This time, as a measure particularly convenient for quantization, the usual Lebesgue measure l on $\mathrm{L}(n)$ should be used,

$$dl(\varphi) = d\varphi^1{}_1 \cdots \varphi^n{}_n.$$

In terms of the two-polar splitting,

$$dl(L,D,R) = P_l(Q)d\mu(L)d\mu(R)dQ^1\cdots dQ^n,$$

where μ, as previously, is the Haar measure on $\mathrm{SO}(n,\mathbb{R})$, and the weight factor P_l is now given by the following expression:

$$P_l = \prod_{a\neq b}\left|(Q^a)^2-(Q^b)^2\right| = \prod_{a\neq b}\left|(Q^a+Q^b)(Q^a-Q^b)\right|.$$

Everything concerning quantization looks in a similar way like previously for affinely-invariant models. For example, expansion of wave functions Ψ with respect to $\mathcal{D}^\alpha(L)$, $\mathcal{D}^\beta(R)$ with $f^{\alpha\beta}(D)$-reduced amplitudes is exactly the same. The difference appears in details concerning the integration procedure, just the weight factor P_l is substituted instead of P. Also, in spite of formal similarities, the particular form of the kinetic energy operator is different,

$$\mathbf{T}^{\mathrm{d.A}}_{\mathrm{int}} = -\frac{\hbar^2}{2I}\mathbf{D}_l + \frac{1}{8I}\sum_{a,b}\frac{(\mathbf{M}^a{}_b)^2}{(Q^a-Q^b)^2} + \frac{1}{8I}\sum_{a,b}\frac{(\mathbf{N}^a{}_b)^2}{(Q^a+Q^b)^2},$$

where now

$$\mathbf{D}_l = \frac{1}{P_l}\sum_a \frac{\partial}{\partial Q^a}P_l\frac{\partial}{\partial Q^a} = \sum_a\frac{\partial^2}{\partial(Q^a)^2} + \sum_a\frac{\partial\ln P_l}{\partial Q^a}\frac{\partial}{\partial Q^a}.$$

Just as previously, the weight factor P_l in the scalar product and first-order differentiations $\partial/\partial Q^a$ may be avoided by rescaling

$$\varphi = \sqrt{P_l}\Psi,$$

but in the resulting differential operator acting on φ also some rather unpleasant potential term appears, i.e.,

$$\widetilde{\mathbf{V}}_l = -\frac{\hbar}{2I}\frac{1}{P_l^2} + \frac{\hbar^2}{4I}\frac{1}{P_l}\sum_a\left(\frac{\partial P_l}{\partial Q^i}\right)^2.$$

It is obvious that without an appropriate potential term $\mathbf{V}$ the geodetic Hamiltonian $\mathbf{T}^{\mathrm{d.A}}$ cannot work in theory of deformable objects because

just as on the classical level it describes only purely scattering, non-bounded motions. Indeed, the above operator

$$\mathbf{T}^{\mathrm{d.A}} = -\frac{\hbar^2}{2I}\Delta^{n^2} = -\frac{\hbar^2}{2I}\sum_{i,A}\frac{\partial^2}{\partial(\varphi^i{}_A)^2}$$

is simply proportional to the usual Laplace operator in $\mathbb{R}^{n^2}$ written in non-typical coordinates.

Therefore, the only realistic applications of the above $\mathbf{T}$ are those as a term of some doubly isotropic Hamiltonian

$$\mathbf{H} = \mathbf{T}^{\mathrm{d.A}} + \mathbf{V}\left(Q^1, \ldots, Q^n\right).$$

Just as previously, due to the double isotropy of the model, the resulting stationary Schrödinger equation

$$\mathbf{H}\Psi = E\Psi$$

splits into the family of equations for partial amplitudes $f^{\alpha\beta}$ depending only on q^a-variables,

$$\mathbf{H}^{\alpha\beta} f^{\alpha\beta} = E^{\alpha\beta} f^{\alpha\beta},$$

where

$$\begin{aligned}\mathbf{H}^{\alpha\beta} f^{\alpha\beta} &= -\frac{\hbar^2}{2I}\mathbf{D}_l f^{\alpha\beta} + \frac{1}{8I}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}{}^a{}_b - \overrightarrow{S^{\alpha}}{}^a{}_b\right)^2}{(Q^a - Q^b)^2} f^{\alpha\beta} \\ &+ \frac{1}{8I}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}{}^a{}_b + \overrightarrow{S^{\alpha}}{}^a{}_b\right)^2}{(Q^a + Q^b)^2} f^{\alpha\beta} + V\left(Q^1, \ldots, Q^n\right) f^{\alpha\beta}.\end{aligned} \tag{7.28}$$

For d'Alembert models, the problem of coverings and multi-valued wave functions looks exactly like in affine theories. Simply $\mathrm{SO}(n, \mathbb{R})$-groups in the two-polar decomposition must be replaced by the coverings $\mathrm{Spin}(n)$. In particular, for $n = 3$ when $\alpha, \beta = s, j = 0, 1/2, 1, \ldots$, everything said above remains true, and $S^{s\,a}{}_b$, $S^{j\,a}{}_b$ are replaced by the standard Wigner matrices of angular momentum, $S^s{}_a$, $S^j{}_a$.

7.29 Usual Wigner matrices $S^j{}_a$

In three dimensions those terms of the affine-affine reduced operator $\mathbf{T}^{\alpha\beta}$ (7.24) which contain the factor $1/32A$ may be written in the following form involving the usual Wigner matrices $S^j{}_a$:

$$\frac{1}{16A}\sum_{a=1}^{3}\frac{(S^s{}_a)^2 f^{sj} - 2S^s{}_a f^{sj} S^j{}_a + f^{sj}(S^j{}_a)^2}{\mathrm{sh}^2\frac{q^b-q^c}{2}}$$
$$-\frac{1}{16A}\sum_{a=1}^{3}\frac{(S^s{}_a)^2 f^{sj} + 2S^s{}_a f^{sj} S^j{}_a + f^{sj}(S^j{}_a)^2}{\mathrm{ch}^2\frac{q^b-q^c}{2}}, \tag{7.29}$$

where in any a-th term of both summations we have obviously $b \neq a$, $c \neq a$, $b \neq c$ (it is clear that it does not matter what is the sequence of b, c).

The same holds for the metric-affine and affine-metric models (7.25), (7.26), with the proviso that the inertial factor A is replaced by α. As mentioned, the last constant-multiplicator terms are respectively

$$\frac{\hbar^2}{2\mu}s(s+1)f^{sj}, \qquad \frac{\hbar^2}{2\mu}j(j+1)f^{sj}.$$

Similarly, in reduced d'Alembert expressions (7.28) the terms with the $1/8I$-factor become for $n = 3$:

$$\frac{1}{4I}\sum_{a=1}^{3}\frac{(S^s{}_a)^2 f^{sj} - 2S^s{}_a f^{sj} S^j{}_a + f^{sj}(S^j{}_a)^2}{(Q^b - Q^c)^2}$$
$$+\frac{1}{4I}\sum_{a=1}^{3}\frac{(S^s{}_a)^2 f^{sj} + 2S^s{}_a f^{sj} S^j{}_a + f^{sj}(S^j{}_a)^2}{(Q^b + Q^c)^2}, \tag{7.30}$$

with the same as previously convention concerning indices a, b, c.

For affinely-invariant geodetic models the bounded state L^2-solutions appear for particular relationships between s and j (α and β) in n dimensions). For the d'Alembert models of kinetic energy this is impossible, an appropriate potential $V\left(Q^1, \ldots, Q^n\right)$ must be always used.

Both the affine and d'Alembert expressions (7.29), (7.30) become particularly simple for the lowest possible values of rotational quantum numbers

s, j, and then there exists some hope for rigorous or at least numerical solutions. Thus, for $s = j = 0$ the corresponding expressions vanish at all, and the resulting Schrödinger equations for f^{00} are purely scalar. For $s = j = 1/2$ we obtain the spinor-spinor state, which is also relatively simple because then ${S^{1/2}}_a = (\hbar/2)\sigma_a$, $\left({S^{1/2}}_a\right)^2 = (\hbar^2/4)I_2$, where, obviously, σ_a are Pauli matrices, and I_2 is the unit 2×2 matrix.

7.30 Two-dimensional case on classical level

In some physical problems also the two-dimensional case $n = 2$ may be physically interesting [49]. And in any case it is mathematically exceptionally simple. This is, so to speak, "pathological" simplicity following from the commutativity of $\mathrm{SO}(2, \mathbb{R})$. Although this exceptional simplicity is rather "exotic" from the point of view of the general n, it may suggest some guiding hints for analysis of this general situation.

The main two-dimensional peculiarity is that

$$\hat{\rho} = \rho = S, \qquad \hat{\tau} = \tau = -V.$$

This is exactly due to the commutativity of $\mathrm{SO}(2, \mathbb{R})$. Because of this, the convenient quantities $\hat{\rho}$, $\hat{\tau}$ are constants of motion for geodetic models and models with doubly-invariant potentials. It was not the case for $n > 2$, where only S, V are constants of motion (for invariant geodetic models and, more generally, for doubly-isotropic models). But it is just the use of $\hat{\rho}$ and $\hat{\tau}$, or equivalently M and N, that simplifies the problem and enables one to perform a partial separation of variables, especially effective on the quantum level. If $n = 2$, the two things coincide, and the problem may be effectively reduced to the Cartan subgroup of diagonal matrices (deformation invariants) even on the classical level.

Let us begin with the classical description. In the two-polar decomposition $\varphi = LDR^{-1}$ we shall use the following parameterization:

$$L = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \qquad R = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix},$$

$$D = \begin{bmatrix} Q^1 & 0 \\ 0 & Q^2 \end{bmatrix} = \begin{bmatrix} \exp q^1 & 0 \\ 0 & \exp q^2 \end{bmatrix}.$$

The splitting $\mathrm{GL}^+(2,\mathbb{R}) = \mathbb{R}^+\mathrm{SL}(2,\mathbb{R})$ is well-suited to coordinates

$$q = \frac{1}{2}\left(q^1 + q^2\right), \qquad x = q^2 - q^1,$$

and their conjugate canonical momenta, respectively,

$$p = p_1 + p_2, \qquad p_x = \frac{1}{2}\left(p_2 - p_1\right).$$

Before using these convenient coordinates, let us express classical kinetic energies in terms of primary variables. First of all, let us notice the obvious fact that the angular velocities of L- and R-rotators are given, respectively, by

$$\chi = \frac{dL}{dt}L^{-1} = L^{-1}\frac{dL}{dt} = \hat{\chi} = \frac{d\alpha}{dt}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$
$$\vartheta = \frac{dR}{dt}R^{-1} = R^{-1}\frac{dR}{dt} = \hat{\vartheta} = \frac{d\beta}{dt}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The corresponding spin and vorticity quantities are given (in canonical representation) by the following expressions:

$$S = \rho = \hat{\rho} = p_\alpha \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad V = -\tau = -\hat{\tau} = p_\beta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where p_α, p_β are, respectively, canonical momenta conjugate to α, β. The corresponding duality pairings are as follows:

$$p_\alpha \frac{d\alpha}{dt} = \frac{1}{2}\mathrm{Tr}(S\chi) = \frac{1}{2}\mathrm{Tr}(\rho\chi) = \frac{1}{2}\mathrm{Tr}(\hat{\rho}\hat{\chi}),$$
$$p_\beta \frac{d\beta}{dt} = \frac{1}{2}\mathrm{Tr}(V\vartheta) = -\frac{1}{2}\mathrm{Tr}(\tau\vartheta) = -\frac{1}{2}\mathrm{Tr}(\hat{\tau}\hat{\vartheta}),$$

where $d\alpha/dt$, $d\beta/dt$ are arbitrary virtual velocities of the variables α, β.

The corresponding classical quantities

$$M = -\hat{\rho} - \hat{\tau}, \qquad N = \hat{\rho} - \hat{\tau}$$

are respectively given by the following expressions:

$$M = \mathsf{m}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = (p_\beta - p_\alpha)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$N = \mathsf{n} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = (p_\beta + p_\alpha) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where

$$\mathsf{m} := p_\beta - p_\alpha, \qquad \mathsf{n} := p_\beta + p_\alpha$$

may be interpreted as canonical momenta conjugate to the corresponding "mixtures" of angles β, α:

$$\gamma := \frac{1}{2}(\beta - \alpha), \qquad \delta := \frac{1}{2}(\beta + \alpha),$$

i.e.,

$$\alpha = \delta - \gamma, \qquad \beta = \delta + \gamma.$$

In fact, one can easily show that

$$\mathsf{m}\dot{\gamma} + \mathsf{n}\dot{\delta} = p_\alpha \dot{\alpha} + p_\beta \dot{\beta}$$

for arbitrary virtual velocities occurring in these formulas, thus,

$$\mathsf{m} = p_\gamma = p_\beta - p_\alpha, \qquad \mathsf{n} = p_\delta = p_\beta + p_\alpha,$$

and conversely,

$$p_\alpha = \frac{1}{2}(\mathsf{n} - \mathsf{m}), \qquad p_\beta = \frac{1}{2}(\mathsf{n} + \mathsf{m}).$$

The previously used magnitudes of S, V become:

$$\|S\| = |p_\alpha| = \frac{1}{2}|\mathsf{n} - \mathsf{m}|, \qquad \|V\| = |p_\beta| = \frac{1}{2}|\mathsf{n} + \mathsf{m}|.$$

For the classical affine-affine kinetic energy (6.11) in Hamiltonian representation we obtain the following expression:

$$\begin{aligned} \mathcal{T}_{\text{int}}^{\text{aff}-\text{aff}} &= \frac{1}{2A}\left(p_1^2 + p_2^2\right) - \frac{B}{2A(A+2B)}p^2 \\ &+ \frac{1}{16A}\frac{\mathsf{m}^2}{\operatorname{sh}^2\frac{q^2-q^1}{2}} - \frac{1}{16A}\frac{\mathsf{n}^2}{\operatorname{ch}^2\frac{q^2-q^1}{2}}; \end{aligned}$$

the meaning of symbols A, B is like previously, and $n = 2$ is substituted to constant factors.

Similarly, for the metrical-affine and affine-metrical models we obtain, respectively,

$$\begin{aligned}
\mathcal{I}_{\text{int}}^{\text{met}-\text{aff}} &= \frac{1}{2\alpha}(p_1^2+p_2^2)+\frac{1}{2\beta}p^2 \\
&+ \frac{1}{16\alpha}\frac{\mathsf{m}^2}{\text{sh}^2\frac{q^2-q^1}{2}}-\frac{1}{16\alpha}\frac{\mathsf{n}^2}{\text{ch}^2\frac{q^2-q^1}{2}}+\frac{1}{8\mu}(\mathsf{n}-\mathsf{m})^2, \\
\mathcal{I}_{\text{int}}^{\text{aff}-\text{met}} &= \frac{1}{2\alpha}(p_1^2+p_2^2)+\frac{1}{2\beta}p^2 \\
&+ \frac{1}{16\alpha}\frac{\mathsf{m}^2}{\text{sh}^2\frac{q^2-q^1}{2}}-\frac{1}{16\alpha}\frac{\mathsf{n}^2}{\text{ch}^2\frac{q^2-q^1}{2}}+\frac{1}{8\mu}(\mathsf{n}+\mathsf{m})^2,
\end{aligned}$$

where the meaning of constants α, β, μ is like previously, but with $n=2$ substituted, thus,

$$\alpha = I+A, \qquad \beta = -\frac{(I+A)(I+A+2B)}{B}, \qquad \mu = \frac{\left(I^2-A^2\right)}{I}.$$

As m and n, or equivalently p_α and p_β, are now constants of motion, it is seen that for geodetic problems and for problems with doubly-isotropic potentials $V\left(q^1,q^2\right)$, e.g., with dilatation-stabilizing ones $V(q)$, everything reduces to the two-dimensional dynamics in variables q^1, q^2 ruled by the effective Hamiltonian obtained by the formal substitution of fixed values p_α, p_β (or m, n) to the above expressions. Moreover, for $\text{SL}(2,\mathbb{R})$-geodetic problems, or for $\text{GL}(2,\mathbb{R})$-problems with separated variables potentials $V(q,x) = V_{\text{dil}}(q)+V_{\text{sh}}(x)$, everything reduces trivially to independent one-dimensional motions. In the above geodetic models it is only the relationship between constant values of m, n that decides whether the motion is oscillatory or unbounded. The first case happens, obviously, when $|\mathsf{n}|>|\mathsf{m}|$; then at large "distances" $\left|q^2-q^1\right|$ the attractive ch^{-2}-term prevails. On the contrary, if $|\mathsf{n}|<|\mathsf{m}|$, one deals with the repulsive case, i.e., with the decaying motion of invariants q^1, q^2. This is the simplest example of the fact mentioned above that affinely-invariant geodetic models admit an open family of bounded (vibrating) and an open family of non-bounded (decaying) motions. Obviously, for general $n>2$ the situation is more complicated because then $M^a{}_b$, $N^a{}_b$ fail to be constants of motion and perform oscillations somehow coupled with those of q^a. Using new variables q, x, p, p_x, we can rewrite

the above models of $\mathcal{T}$ in the following forms:

$$\begin{aligned}
\mathcal{T}_{\text{int}}^{\text{aff}-\text{aff}} &= \frac{p^2}{4(A+2B)} + \frac{p_x^2}{A} + \frac{(p_\alpha - p_\beta)^2}{16A\text{sh}^2\frac{x}{2}} - \frac{(p_\alpha + p_\beta)^2}{16A\text{ch}^2\frac{x}{2}},\\
\mathcal{T}_{\text{int}}^{\text{met}-\text{aff}} &= \frac{p^2}{4(I+A+2B)} + \frac{p_x^2}{I+A}\\
&+ \frac{(p_\alpha - p_\beta)^2}{16(I+A)\text{sh}^2\frac{x}{2}} - \frac{(p_\alpha + p_\beta)^2}{16(I+A)\text{ch}^2\frac{x}{2}} + \frac{Ip_\alpha^2}{I^2-A^2},\\
\mathcal{T}_{\text{int}}^{\text{aff}-\text{met}} &= \frac{p^2}{4(I+A+2B)} + \frac{p_x^2}{I+A}\\
&+ \frac{(p_\alpha - p_\beta)^2}{16(I+A)\text{sh}^2\frac{x}{2}} - \frac{(p_\alpha + p_\beta)^2}{16(I+A)\text{ch}^2\frac{x}{2}} + \frac{Ip_\beta^2}{I^2-A^2}.
\end{aligned}$$

In the special case $n = 2$, it is easily seen that on the level of variables q, x all these geodetic models have identical dynamics. The difference appears only on the level of angular variables α, β. And, just as for the general n, the same is true if we introduce to Hamiltonians some doubly-isotropic potentials $V(q, x)$. In particular, this is true for dilatation-stabilizing potentials $V(q)$, i.e., in a sense, for geodetic invariant models on $\text{SL}(2, \mathbb{R})$ (incompressible bodies).

7.31 Quantization of two-dimensional models

Let us now turn to quantization. The Haar measure λ on $\text{GL}(2, \mathbb{R})$ is given by the following expression:

$$d\lambda\left(\alpha; q^1, q^2; \beta\right) = \left|\text{sh}\left(q^1 - q^2\right)\right| d\alpha d\beta dq^1 dq^2,$$

i.e.,

$$d\lambda\left(\alpha; q, x; \beta\right) = \left|\text{sh}x\right| d\alpha d\beta dq dx, \qquad P = \left|\text{sh}x\right|.$$

The Peter-Weyl expansion with respect to the L, R-factors of the two-polar splitting is just the usual double Fourier series:

$$\Psi\left(\alpha; q, x; \beta\right) = \sum_{m,n\in\mathbb{Z}} f^{mn}(q, x) e^{im\alpha} e^{in\beta}.$$

The reduced kinetic Hamiltonian corresponding to $\mathcal{T}^{\mathrm{aff}-\mathrm{aff}}_{\mathrm{int}}$ is as follows:

$$\begin{aligned}\mathbf{T}^{mn}f^{mn} &= -\frac{\hbar^2}{A}\mathbf{D}_x f^{mn} - \frac{\hbar^2}{4(A+2B)}\frac{\partial^2 f^{mn}}{\partial q^2} \\ &+ \frac{\hbar^2(n-m)^2}{16A\mathrm{sh}^2\frac{x}{2}}f^{mn} - \frac{\hbar^2(n+m)^2}{16A\mathrm{ch}^2\frac{x}{2}}f^{mn},\end{aligned}$$

where

$$\mathbf{D}_x f^{mn} = \frac{1}{|\mathrm{sh}x|}\frac{\partial}{\partial x}\left(|\mathrm{sh}x|\frac{\partial f^{mn}}{\partial x}\right).$$

For the metric-affine and affine-metric models $\mathcal{T}^{\mathrm{met}-\mathrm{aff}}_{\mathrm{int}}$, $\mathcal{T}^{\mathrm{aff}-\mathrm{met}}_{\mathrm{int}}$ we obtain, respectively, the following expressions:

$$\begin{aligned}\mathbf{T}^{mn}f^{mn} &= -\frac{\hbar^2}{I+A}\mathbf{D}_x f^{mn} - \frac{\hbar^2}{4(I+A+2B)}\frac{\partial^2 f^{mn}}{\partial q^2} \\ &+ \frac{\hbar^2(n-m)^2}{16(I+A)\mathrm{sh}^2\frac{x}{2}}f^{mn} - \frac{\hbar^2(n+m)^2}{16(I+A)\mathrm{ch}^2\frac{x}{2}}f^{mn} + \frac{I\hbar^2m^2}{I^2-A^2}f^{mn},\end{aligned}$$

$$\begin{aligned}\mathbf{T}^{mn}f^{mn} &= -\frac{\hbar^2}{I+A}\mathbf{D}_x f^{mn} - \frac{\hbar^2}{4(I+A+2B)}\frac{\partial^2 f^{mn}}{\partial q^2} \\ &+ \frac{\hbar^2(n-m)^2}{16(I+A)\mathrm{sh}^2\frac{x}{2}}f^{mn} - \frac{\hbar^2(n+m)^2}{16(I+A)\mathrm{ch}^2\frac{x}{2}}f^{mn} + \frac{I\hbar^2n^2}{I^2-A^2}f^{mn}\end{aligned}$$

with the same meaning of symbols as previously. It is seen that in all these expressions the complete separation between dilatational and incompressible motion is very effectively described in analytical terms just due to the use of coordinates q, x. Obviously, for geodetic Hamiltonians on $\mathrm{GL}(2,\mathbb{R})$ the energy spectrum is continuous (and classical trajectories are unbounded; in a sense equivalent facts) because dilatational motion is free. As in the general case, this fact is physically avoided by introducing to the Hamiltonian some dilatation-stabilizing potential $V_{\mathrm{dil}}(q)$. On the quantum level the simplest possible model is the potential well.

This is, in a sense, reduction to the geodetic quantum problem on $\mathrm{SL}(2,\mathbb{R})$. Obviously, the problem with $V_{\mathrm{dil}}(q)$ remains explicitly separable. It remains so also for a more general class of doubly isotropic potentials, e.g., for ones explicitly splitting,

$$V(q,x) = V_{\mathrm{dil}}(q) + V_{\mathrm{sh}}(x),$$

but perhaps also for more general ones. Solutions of the corresponding stationary Schrödinger equations may be sought in the following form:

$$f^{mn}(q,x) = \varphi^{mn}(q)\chi^{mn}(x);$$

the problem reduces then to one-dimensional Schrödinger equations for φ^{mn} and χ^{mn}. And now, in the special two-dimensional case, it is explicitly seen that there exists a discrete spectrum (bounded situations) for χ-functions, i.e., for the isochoric $\mathrm{SL}(2,\mathbb{R})$-problem, even in the purely geodetic case without any potential $V_x(x)$. And this is true in spite of the non-compactness of the $\mathrm{SL}(2,\mathbb{R})$-configuration space. Everything depends on the mutual relationship between "rotational" quantum numbers m, n. If $|n+m| > |n-m|$, the attractive ch^{-2}-term prevails at large "distances" $|x| \to \infty$ and the spectrum is discrete. In the opposite case, if $|n+m| < |n-m|$, it is continuous.

For the affine-affine geodetic model on $\mathrm{SL}(2,\mathbb{R})$, the total spectrum (total in the sense of solutions for all possible $m, n \in \mathbb{Z}$) is not bounded from below; this might seem undesirable. For the metric-affine and affine-metric geodetic problems on $\mathrm{SL}(2,\mathbb{R})$, the spectrum may be bounded from below (and so is the corresponding kinetic energy). Everything depends on the mutual relationship between inertial constants I, A, B, which play the role of some controlling parameters.

7.32 Two-dimensional d'Alembert models

For comparison, let us quote a few corresponding formulas for the "usual" d'Alembert model in two dimensions. We restrict ourselves to the doubly-isotropic model. The classical kinetic Hamiltonian may be expressed as follows:

$$\mathcal{T}^{\mathrm{d.A}}_{\mathrm{int}} = \frac{1}{2I}\left(P_1^2 + P_2^2\right) + \frac{1}{4I}\frac{\mathsf{m}^2}{\left(Q^1 - Q^2\right)^2} + \frac{1}{4I}\frac{\mathsf{n}^2}{\left(Q^1 + Q^2\right)^2},$$

with the same meaning of symbols as previously. Let us stress that Q^a are diagonal elements of D, and now the variables $q^a = \ln Q^a$ would be completely useless. The quantity P_l is given simply by the following expression:

$$P_l = \left|\left(Q^1\right)^2 - \left(Q^2\right)^2\right| = \left|\left(Q^1 + Q^2\right)\left(Q^1 - Q^2\right)\right|,$$

and the usual Lebesgue measure on $\mathrm{L}(2,\mathbb{R}) \simeq \mathbb{R}^4$ is expressed as follows:

$$dl\left(\alpha; Q^1, Q^2; \beta\right) = P_l\left(Q^1, Q^2\right) d\alpha d\beta dQ^1 dQ^2.$$

As mentioned, geodetic models are non-physical (and, by the way, the above coordinates would be completely artificial for them). There is, however, a class of physically reasonable doubly isotropic potentials $V\left(Q^1, Q^2\right)$ for which the corresponding Hamiltonians $H = \mathcal{T} + V$ describe integrable systems admitting solutions in terms of separation of variables. This fact is obvious when, instead of Q^1, Q^2, the $(\pi/4)$-rotated coordinates Q^+, Q^- on the plane of deformation invariants are used,

$$Q^{\pm} := \frac{1}{\sqrt{2}}\left(Q^1 \pm Q^2\right).$$

The polar and elliptic coordinates on the (Q^+, Q^-)-plane are also convenient,

$$Q^+ = r\cos\varphi, \qquad Q^- = r\sin\varphi$$

and

$$Q^+ = \mathrm{ch}\rho\cos\lambda, \qquad Q^- = \mathrm{sh}\rho\sin\lambda.$$

There exist physically reasonable potentials V for which the corresponding Hamiltonian problems are separable (thus, obviously, integrable) in coordinates (Q^+, Q^-), (r, φ), or (ρ, λ). There are also interesting superintegrable (degenerate) models separable simultaneously in two or even three of the above coordinate systems.

On the quantized level the reduced Schrödinger equation has the following form:

$$\mathbf{H}^{mn} f^{mn} = E^{mn} f^{mn},$$

where

$$\mathbf{H}^{mn} f^{mn} = \mathbf{T}^{mn} f^{mn} + \mathbf{V}\left(Q^1, Q^2\right) f^{mn}$$

$$= -\frac{\hbar^2}{2I}\mathbf{D}_l f^{mn} + \frac{\hbar^2 m^2}{4I(Q^1 - Q^2)^2} f^{mn} + \frac{\hbar^2 n^2}{4I(Q^1 + Q^2)^2} f^{mn} + \mathbf{V}\left(Q^1, Q^2\right) f^{mn}.$$

Obviously,

$$\mathbf{D}_l f = \frac{1}{P_l}\sum_{a=1}^{2} \frac{\partial}{\partial Q^a}\left(P_l \frac{\partial f}{\partial Q^a}\right).$$

Everything said above about separability of the classical problems remains true on the quantized level. Again the coordinate systems (Q^+, Q^-), (r, φ), (ρ, λ) are crucial.

7.33 Hamiltonian systems on $\mathrm{U}(n)$

To finish these quantization remarks let us mention briefly about Hamiltonian systems on $\mathrm{U}(n)$, i.e., in a sense, affine systems with "compactified deformation invariants" (6.11). The resulting kinetic energy operator has the following form:

$$\begin{aligned}\mathbf{T} &= -\frac{\hbar^2}{2A}\mathbf{D}_U + \frac{\hbar^2 B}{2A(A+nB)}\frac{\partial^2}{\partial q^2} \\ &+ \frac{1}{32A}\sum_{a,b}\frac{(\mathbf{M}^a{}_b)^2}{\sin^2\frac{q^a-q^b}{2}} + \frac{1}{32A}\sum_{a,b}\frac{(\mathbf{N}^a{}_b)^2}{\cos^2\frac{q^a-q^b}{2}},\end{aligned}$$

where

$$\begin{aligned}\mathbf{D}_U &= \frac{1}{P_U}\sum_a \frac{\partial}{\partial q^a} P_U \frac{\partial}{\partial q^a} = \sum_a \frac{\partial^2}{\partial (q^a)^2} + \sum_a \frac{\partial \ln P_U}{\partial q^a}\frac{\partial}{\partial q^a}, \\ P_U &= \prod_{a\neq b}\left|\sin(q^a - q^b)\right|.\end{aligned}$$

The Haar measure is given by the following expression:

$$d\lambda_U(L,D,R) = P_U d\mu(L) d\mu(R) dq^1 \cdots dq^n,$$

where μ, as previously, denotes the Haar measure on $\mathrm{SO}(n,\mathbb{R})$.

Obviously, $\mathrm{U}(n)$ is compact, thus, all classical trajectories for geodetic models are bounded and the corresponding quantum spectrum is discrete. Nevertheless, more general models with doubly-isotropic potentials,

$$\mathbf{H} = \mathbf{T} + \mathbf{V}\left(q^1, \ldots, q^n\right),$$

may be also of physical interest.

The problem splits again, just as in the $\mathrm{GL}(n,\mathbb{R})$-case, into the family of reduced problems resulting from the Fourier analysis on $\mathrm{SO}(n,\mathbb{R})$ performed both in the L- and R-variables:

$$\mathbf{H}^{\alpha\beta} f^{\alpha\beta} = E^{\alpha\beta} f^{\alpha\beta},$$

where

$$\begin{aligned}\mathbf{H}^{\alpha\beta}f^{\alpha\beta} &= -\frac{\hbar^2}{2I}\mathbf{D}_U f^{\alpha\beta} - \frac{\hbar^2}{2\beta}\frac{\partial^2 f^{\alpha\beta}}{\partial q^2} + \frac{1}{32A}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta a}}_b - \overrightarrow{S^{\alpha a}}_b\right)^2}{\sin^2\frac{q^a-q^b}{2}}f^{\alpha\beta}\\ &+ \frac{1}{32A}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta a}}_b + \overrightarrow{S^{\alpha a}}_b\right)^2}{\cos^2\frac{q^a-q^b}{2}}f^{\alpha\beta} + \mathbf{V}\left(q^1,\ldots,q^n\right)f^{\alpha\beta},\end{aligned}$$

with the same meaning of symbols as previously.

Just as in the $\mathrm{GL}(n,\mathbb{R})$-models, particularly simple are physical dimensions $n = 2, 3$. The former one has also certain very peculiar features and admits simple calculations based on integrable models and separability techniques. Namely, $\mathbf{H}^{mn}$ acts as follows:

$$\begin{aligned}\mathbf{H}^{mn}f^{mn} &= -\frac{\hbar^2}{A}\mathbf{D}_x f^{mn} + \frac{\hbar^2(n-m)^2}{16A\sin^2\frac{x}{2}}f^{mn} + \frac{\hbar^2(n+m)^2}{16A\cos^2\frac{x}{2}}f^{mn}\\ &+ \mathbf{V}_x(x)f^{mn} - \frac{\hbar^2}{4(A+2B)}\frac{\partial^2 f^{mn}}{\partial q^2} + \mathbf{V}_q(q)f^{mn},\end{aligned}$$

where, obviously, the Haar measure has the following form:

$$d\lambda_U\left(\alpha;q,x;\beta\right) = |\sin x| d\alpha d\beta dq dx,$$

and

$$\mathbf{D}_x f = \frac{1}{|\sin x|}\frac{\partial}{\partial x}\left(|\sin x|\frac{\partial f}{\partial x}\right).$$

The problem also separates, in particular, for geodetic problems, $V = 0$, or for potentials of the above-mentioned form:

$$V(q,x) = V_{\mathrm{dil}}(q) + V_{\mathrm{sh}}(x).$$

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